Math Identities Proofs

through Derivatives (2023)

A Reference Guide for Higher Mathematics

[Bill Liam East]

“Usually mathematical proofs lead to some form of rehab.”

– Professor James Gilbert

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(Hyperlinked when downloaded)

Mathematical Basics

Arithmetic Rules

Commutative Properties

Additive $a + b = b + a$ Multiplicative $a \cdot b = b \cdot a$

Associative Properties

Additive $(a + b) + c = a + (b + c)$ Multiplicative $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

Distributive Property

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

Identities

Additive $a + 0 = a = 0 + a$ Multiplicative $a \cdot 1 = a = 1 \cdot a$

Inverse Properties

Additive $a + (-a) = 0$ Multiplicative $a \cdot a^{-1} = 1$

Transitive Property

If two quantities are equal with a third, then they are equal with each other. Commonly, if $a = b$ and $b = c$, then $a = c$.

Jump to Squeeze Theorem

Distribution of Negatives

One negative distributes: $\frac{-1}{1} = \frac{1}{-1} = -\frac{1}{1} = -1$ Two negatives cancel: $\frac{-1}{-1} = -\left(-\frac{1}{1}\right) = 1$

Jump to Derivative Quotient Rule

Reciprocal Rule of Division

For expressions $a = \frac{1}{1/a}$ For evaluating fractions $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$

Jump to L'Hôpital's Rule

Plus or Minus Notation

If $\pm$ is used on one side of an equation, then the equation has two solutions

$$x = a \pm b \leftrightarrow x = a + b \text{ and } x = a - b$$

If $\pm$ is used on two sides of an equation, then when it is one sign on one side, it is the same on the other

$$\pm x = a \pm b \leftrightarrow x = a + b \text{ and } -x = a - b$$

If $\pm$ and $\mp$ used, then when one is positive, the other is negative

$$\pm x = a \mp b \leftrightarrow x = a - b \text{ and } -x = a + b$$

If $\pm$ is used in an exponent, it is used to indicate multiplication or division and the same rules above apply

$$x \cdot y^{\pm 1} = a \pm b \leftrightarrow x \cdot y = a + b \text{ and } x/y = a - b$$

Number Types

Number Sets and Definitions

Real All common numbers and multiples of one. It is represented as $\mathbb{R}$.

Integer Real whole numbers that can be expressed without using a fraction (-2, -1, 0, 1, 2). It is represented as $\mathbb{Z}$.

Natural Positive real integers. It is argued as including zero and is represented as $\mathbb{N}$.

Rational   Any number that can be represented as a fraction inclusive of only natural numbers, i.e. $1/2$, $2/1$, 1.2.

Irrational   Any number that cannot be represented as a fraction. They are non-terminating, i.e. $\sqrt{2}, e, \pi, \phi$.

Imaginary   All numbers that are multiples of i.

Complex   Numbers comprised of both real and imaginary numbers. It is represented as $\mathbb{C}$.

Repeating Values

Series of Nine

Repeating values can always be represented as a fraction with the value over an equal number of 9's to the value.

$$.\overline{2} = \frac{2}{9} \qquad .\overline{137} = \frac{137}{999} \qquad .\overline{142857} = \frac{142857}{999999} = \frac{1}{7}$$

Proof Nine Repeating Equals One

Let $.\overline{9}$ equal a variable, multiply by ten, separate $9.\overline{9}$ into 9 and $.\overline{9}$, substitute $.\overline{9}$ for x, subtract x, divide by nine.

$$x = .\overline{9} \qquad 10 \cdot x = 9.\overline{9} = 9 + .\overline{9} \qquad 10 \cdot x = 9 + x \qquad 9 \cdot x = 9 \qquad x = 1$$

Orders of Ten

Where a number starts repeating is a matter of orders of ten, which can be isolated.

$$433.\overline{3} = 400 + \frac{3}{9} \cdot 10^2 = \frac{1300}{3} \qquad .35\overline{7} = \frac{35}{100} + \frac{7}{9} \cdot 10^{-2} = \frac{161}{450} \qquad 3.8\overline{3} = \frac{1}{2} + \frac{30}{9} = \frac{23}{6}$$

Angular Units

A full rotation in one angular direction from starting point back to the starting point is 360°. It is converted as follows:

Degrees (D)	Gradians	Radians	Milliradians	Minutes of arc	Seconds of arc
$D \cdot 1$	$D \cdot \frac{10}{9}$	$D \cdot \frac{\pi}{180°}$	$D \cdot \frac{1000\pi}{180°}$	$D \cdot 60$	$D \cdot 3600$
360°	400g	$2 \cdot \pi$	$2000 \cdot \pi$	21600	1296000

Indeterminate Forms

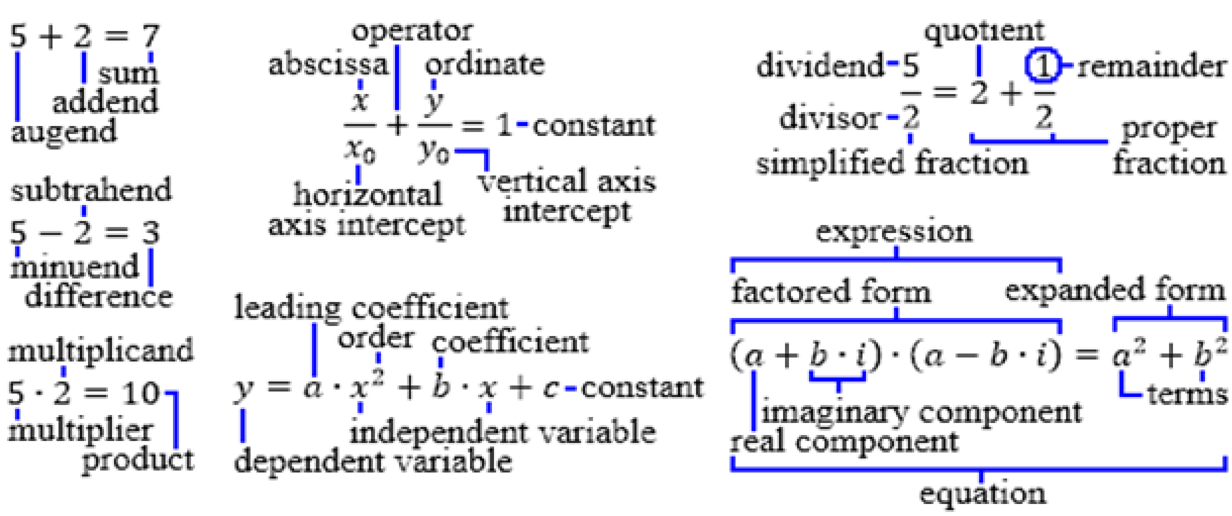
Undefined forms with no number value, they do not exist and therefore cannot be operated on arithmetically.

$$\frac{0}{0} \qquad \frac{\infty}{\infty} \qquad 0 \cdot \infty \qquad \infty - \infty \qquad 0^0 \qquad 1^\infty \qquad \infty^0$$

Jump to L'Hôpital's Rule

Functions

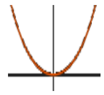
Terminology



Jump to Prime Quadratic Equations Factored (Completing the Square)

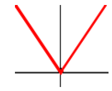
Function Definitions

Even – smmetric about y-axis



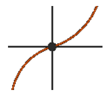
$$f(-x) = f(x)$$

Absolute Value – magnitude; non-negative



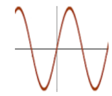
$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Odd – symmetric about origin



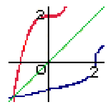
$$f(-x) = -f(x)$$

Periodic – repeated segment (P)



$$f(x+P) = f(x)$$

Inverse – reflected about $y = x$



$$f\big(g(x)\big) = g\big(f(x)\big) = x, \forall x \in \big(x_o, x_f\big)$$

Jump to Imaginary Units Exponentiated to Integers
Jump to Limits of Trig Functions

Function Transformations

Given $a \cdot f(b \cdot x + c) + d$,

- If $|a| > 1$, f stretches vertically.
- If $|a| < 1$, f compresses vertically.
- If $a < 0$, f flips vertically.
- If $c > 0$, f shifts to the left.
- If $c < 0$, f shifts to the right.
- If $|b| > 1$, f compresses horizontally.
- If $|b| < 1$, f stretches horizontally.
- If $b < 0$, f flips vertically.
- If $d > 0$, f shifts upwards.
- If $d < 0$, f shifts downwards.

Mathematical and Logical Operators

Sy.	Meaning	Example	Translation (in much simpler terms)
$\in$	As an element of	$1 \in \mathbb{N}$	One is a natural number
$\mathbb{R}^3$	Dimensions of set $\mathbb{R}$	$\{x, y, z\} \in \mathbb{R}^3$	Three directions existing in real space
$\sum$	The sum of	$\Sigma\mathbb{N} = 1 + 2 + 3 + 4 \ldots$	All natural numbers added together
Π	The product of	$\Pi\mathbb{N} = 1 \cdot 2 \cdot 3 \cdot 4 \ldots$	All natural numbers multiplied together
!	Factorial	$5! = \Pi 5$	$1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$
$[a, b]$	Domain/range inclusive	$[0,1]$	All numbers from zero to one
(a, b)	Domain/range exclusive	$(0,1)$	All numbers between zero and one
$\{a, b\}$	Any size list of values	$\{4, 5, 1, 3, 2\}$	Produces same size list when operated for each value
Mod	Modulus	$4 \bmod 3 = 1$	The remainder of 4/3 (periodic: $-4/3 = 2$)
$\forall$	For all	$\forall n \in \mathbb{R}: 2 \cdot n = n + n$	For all real numbers the following holds
$\exists$	There exists	$\exists r \in \mathbb{R}: 0 \cdot r = 0$	Any real number satisfies the following
$\exists!$	There exists uniquely	$\exists! n \in \mathbb{N}: (n + 1 = 2)$	One natural number satisfies the following
$\equiv$	Is equal by definition to	$c \equiv 299{,}792{,}458 \text{ m/s}$	The speed of light is exactly defined as this value
$\therefore$	Therefore	$x - y = 0 \therefore x = y$	This statement is true therefore that is also true
$\wedge$	And	$x > 2 \wedge x \neq 3$	X is greater than 2 and not equal to 3
$\vee$	Or	$x = 2 \vee x = -2$	X is equal to 2 or –2 or both
$\parallel$	Is parallel to	$a \cdot x + b \parallel c \cdot x + d$	The two lines are in the same direction
$\perp$	Is perpendicular to	$a \cdot x + b \perp c \cdot x + d$	The two lines together form right angles
x_0	Point of origin	$y(x) = y_0 + m \cdot x$	Starting point, in most cases, the y-intercept
$\bar{x}$	Average value	$If\ x = \{4,5,3\}, \bar{x} = 4$	The average value of all given values
$\vec{v}$	Vector quantity	$\vec{v} = x \cdot \hat{\imath} + y \cdot \hat{\jmath}$	Symbol of component directions and magnitude
$\Vert\vec{v}\Vert$	Vector magnitude	$\Vert\vec{v}\Vert = \sqrt{\vec{v} \cdot \vec{v}}$	Absolute value of a vector length
x^*	Complex conjugate	$(3 + 5 \cdot i)^* = 3 - 5 \cdot i$	The sign of the imaginary (or second) part is reversed
X^T	Transpose	${A_{ij}}^T = A_{ji}$	Matrix columns and rows are interchanged
$X^\dagger$	Conjugate transpose	$(X^*)^T = X^\dagger$	Matrix both complex conjugated and transposed

Complex Numbers

Imaginary Number Definition

Square roots of negative real numbers fail to yield a real result. The imaginary unit, $\sqrt{-1}$, satisfies the equation $i^2 = -1$

Jump to Complex Number System

Imaginary Units Exponentiated to Integers

Powers of i always reduce using a modulus of 4: $i^x = i^{(x \bmod 4)}$.

- Even powers simplify into real numbers. ($i^{-2} = i^2 = i^6 = -1$ and $i^{-4} = i^0 = i^4 = 1$).
- Odd powers simplify into imaginary numbers. ($i^{-3} = i^1 = i^5 = i$ and $i^{-1} = i^3 = i^7 = -i$).
- It follows that imaginary numbers with real integer exponents are a periodic series of $i, -1, -i, 1$.

Jump to Complex Unit Circle

Complex Conjugation

Using the representation $z = x + y \cdot i$, it is the sign reversal of the imaginary part

$$z^* = x - y \cdot i$$

Magnitude of Complex Numbers

$$|z|^2 = z \cdot z^* = x^2 + y^2$$

Complex Division

Division uses multiplication of the complex conjugate in both the divisor and dividend of any term

$$\frac{a + b \cdot i}{c + d \cdot i} = \frac{a + b \cdot i}{c + d \cdot i} \cdot \frac{a - b \cdot i}{c - d \cdot i} = \frac{a \cdot c + b \cdot d + (b \cdot c - a \cdot d) \cdot i}{c^2 + d^2}$$

Exponentiation

Exponents

Universal Properties

Exponentiation is repeated multiplication. An exponent is the number of times timed, for example, x^3 has an exponent of 3.

- Exponents do not commute: $x^a \neq a^x$
- Exponents do not associate: $x^{(a^b)} \neq (x^a)^b$
- Factoring is the condensing of terms: x^3
- The reverse of factoring is expansion: $x \cdot x \cdot x$
- Expressions factorable into integers are composite, otherwise it is prime.

Negative Exponents

$$x^{-a} = \frac{1}{x^a}$$

Deductive Logic

Multiplication is reversible with division, therefore exponential reduction yields inverses once reaching negatives.

$x^3 = 1 \cdot x \cdot x \cdot x$	$x^2 = 1 \cdot x \cdot x$	$x^1 = 1 \cdot x$	$x^0 = 1$	$x^{-1} = 1/x$	$x^{-2} = 1/(x \cdot x)$
$3^3 = 27$	$3^2 = 9$	$3^1 = 3$	$3^0 = 1$	$3^{-1} = 1/3$	$3^{-2} = 1/9$

Power Rule

$$x^a \cdot x^{\pm b} = x^{a \pm b}$$

Jump to Product and Quotient Rules of Logarithms
Jump to Derivative of e Exponentiated

Deductive Logic

x^a is x multiplied a times, x^b is x multiplied b times, and they are multiplied to each other so the exponents are added

$$2^3 \cdot 2^{-5} = 2^{3-5} = \frac{\cancel{2 \cdot 2 \cdot 2}}{2 \cdot 2 \cdot \cancel{2 \cdot 2 \cdot 2}} = 2^{-2} = \frac{1}{2^2} = \frac{1}{4}$$

Repeated Exponentials

Evaluate starting from the highest exponent down with any specifics inside parenthesized terms first.

$$x^{a^{b^{c^d}}} = x^{\left(a^{\left(b^{\left(c^{(d)}\right)}\right)}\right)}$$

Power Distribution

$$(x^a \cdot y^{\pm b})^c = x^{a \cdot c} \cdot y^{\pm b \cdot c}$$

Jump to Power Rule of Logarithms
Jump to Analytic Trig Inverses

Deductive Logic

Expand into $(x^a)^c \cdot (y^{\pm b})^c$, and distribute in accordance with the terms being multiplied c times.

Example

Given $3 \cdot x - 5 \cdot y = 2$, evaluate $8^x / 32^y$

$$\frac{8^x}{32^y} = \frac{(2^3)^x}{(2^5)^y} = \frac{2^{3 \cdot x}}{2^{5 \cdot y}} = 2^{3 \cdot x - 5 \cdot y} = 2^2 = 4$$

Real Radical Exponents

For radical exponents, the denominator symbolizes a root, such as a square root if it is 2.

$$x^{p/q} = \left(\sqrt[q]{x}\right)^p = \sqrt[q]{x^p}, \forall x \geq 0$$

Root Expansion and Factoring

$$\sqrt[n]{x \cdot y^{\pm 1}} = \sqrt[n]{x} \cdot \sqrt[n]{y^{\pm 1}}, x > 0 \ \vee \ y > 0$$

Either x or y must be positive because complex roots cannot be "extracted" from a real root and vice versa.

Jump to Trig Half Angle Identities

Even Roots of Even Powers

$$\sqrt[n]{x^n} = |x|, n \in 2\mathbb{Z}$$

Scientific Notation

Scientific notation displays all numbers as multiples of ten to a power in accordance with the leading digit.

$299792458 = 2.99792458 \cdot 10^8 = 2.99792458\text{E}8$

$0.0820574 = 8.20574 \cdot 10^{-2} = 8.20574\text{E}{-2}$

Polynomials

Cancellation

In a factored function, like terms in both a numerator and denominator as multipliers reduce to 1.

$$\frac{a(x) \cdot f(x)}{a(x) \cdot g(x)} = \frac{f(x)}{g(x)}$$

$$\frac{a(x) \cdot f(x)}{-a(x) \cdot g(x)} = -\frac{f(x)}{g(x)}$$

Examples

$$\frac{(x-4)}{(x-4)} \cdot \frac{f(x)}{g(x)} = \frac{f(x)}{g(x)}$$

$$\frac{(x-4)}{(4-x)} \cdot \frac{f(x)}{g(x)} = -\frac{f(x)}{g(x)}$$

Quadratic Equations Definition

Second degree trinomial equations in the form $a \cdot x^2 + b \cdot x + c = 0$, or more generally, $y = a \cdot x^2 + b \cdot x + y_0$.

Jump to Universal Properties of Parabolas
Jump to The Golden Ratio φ
Jump to Analytic Trig Inverses

Composite Quadratic Equations Factored

$$a \cdot c \cdot x^2 \pm (a \cdot d + b \cdot c) \cdot x + b \cdot d = (a \cdot x \pm b) \cdot (c \cdot x \pm d)$$

Prime Quadratic Equations Factored (Completing the Square)

The leading coefficient must be 1, otherwise the equation must be multiplied for it to be 1. $(b/2)^2$ is added for factoring.

$$x^2 \pm b \cdot x + y_0 = y \therefore x^2 \pm b \cdot x + \left(\frac{b}{2}\right)^2 = y - y_0 + \left(\frac{b}{2}\right)^2 \therefore \frac{(2 \cdot x \pm b)^2}{4} = y - y_0 + \frac{b^2}{4}$$

Jump to Universal Properties of Circles
Jump to Universal Properties of Ellipses
Jump to Universal Properties of Parabolas

Quadratic Formula

The (more general) solutions of $y = a \cdot x^2 + b \cdot x + y_0$ are:

$$x = \frac{-b \pm \sqrt{b^2 + 4 \cdot a \cdot (y - y_0)}}{2 \cdot a}, a \neq 0$$

If $y = 0$ as normally used, then y_0, or c, is the y-intercept, and the resulting values of x are the x-intercepts.

Jump to Universal Properties of Parabolas
Jump to The Golden Ratio φ
Jump to Analytic Trig Inverses

Proof

Subtract y_0 from both sides, then multiply both sides by $4 \cdot a$

$$4 \cdot a^2 \cdot x^2 + 4 \cdot a \cdot b \cdot x = 4 \cdot a \cdot (y - y_0)$$

Add b^2 to both sides and factor the left side, similar to completing the square.

$$(2 \cdot a \cdot x + b)^2 = b^2 + 4 \cdot a \cdot (y - y_0)$$

Take the square root of both sides.

$$2 \cdot a \cdot x + b = \pm\sqrt{b^2 + 4 \cdot a \cdot (y - y_0)}$$

Subtract b from both sides, then divide both sides by $2 \cdot a$.

$$x = \frac{-b \pm \sqrt{b^2 + 4 \cdot a \cdot (y - y_0)}}{2 \cdot a}, a \neq 0$$

If the coefficient a equals zero, the equation is linear, to which this does not apply.

Discriminant of a Quadratic Formula

$$\Delta = b^2 - 4 \cdot a \cdot y_0$$

- If $\Delta > 0$, the equation has two real factorable roots
- If $\Delta = 0$, the equation has one factorable root squared
- If $\Delta < 0$, the equation has two complex factorable roots

Jump to Universal Properties of Parabolas

Sum and Difference of Powers

Sum of Powers for Real Roots (Odd Exponents Only)

$$a^n + b^n = (a + b)(a^{n-1} - a^{n-2} \cdot b + a^{n-3} \cdot b^2 - \cdots + a^2 \cdot b^{n-3} - a \cdot b^{n-2} + b^{n-1}), \forall 2 \cdot n - 1 \in \mathbb{N}$$

Difference of Powers for Real Roots

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}\cdot b + a^{n-3}\cdot b^2 + \cdots + a^2\cdot b^{n-3} + a\cdot b^{n-2} + b^{n-1}), \forall n \in \mathbb{N}$$

Sum and Difference of Squares and Cubes

$$a^2 + b^2 = (a + i\cdot b)(a - i\cdot b) \qquad a^2 - b^2 = (a+b)(a-b) \qquad a^3 \pm b^3 = (a \pm b)(a^2 \mp a\cdot b + b^2)$$

Other Sums and Differences of Interest

$$a^4 - b^4 = (a + i\cdot b)(a - i\cdot b)(a+b)(a-b)$$

$$a^6 + b^6 = (a^2 + b^2)(a^4 - a^2\cdot b^2 + b^4)$$

$$a^{10} + b^{10} = (a^2 + b^2)(a^8 - a^6\cdot b^2 + a^4\cdot b^4 - a^2\cdot b^6 + b^8)$$

Pascal's Triangle

Each number in the given 'triangle' below the first is the sum of the numbers diagonally above it.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

Binomials

Binomial Coefficient

$$\binom{n}{k} = \frac{n!}{(n-k)!\cdot k!} = \frac{n\cdot(n-1)\cdot(n-2)\ldots(n-k+1)}{k!}$$

Binomial Theorem

The coefficients in the expanded form match the line n+1 down from the top on Pascal's triangle.

$$(x+y)^n = \binom{n}{0}\cdot x^n + \binom{n}{1}\cdot x^{n-1}\cdot y + \binom{n}{2}\cdot x^{n-2}\cdot y^2 + \cdots + \binom{n}{n}\cdot y^n, \forall n \in \mathbb{Z}$$

Jump to Derivative Power Rule

Binomials Squared and Cubed

$$(a \pm b)^2 = a^2 \pm 2\cdot a\cdot b + b^2 \qquad (a \pm b)^3 = a^3 \pm 3\cdot a^2\cdot b + 3\cdot a\cdot b^2 \pm b^3$$

Polynomial Long Division

Given the rational function $N(x)/D(x)$, the division results in the equality $N(x) = D(x)\cdot Q(x) + R(x)$., or

$$\begin{array}{r} Q(x) \; + R(x) \\ N(x)\overline{)D(x)} \end{array}$$

Solving by Example

Given $(x-2)/(x^3 + 2\cdot x^2 + 12)$, express in the following form with missing (zero coefficient) terms included:

$$x - 2\overline{)x^3 + 2\cdot x^2 + 0\cdot x + 12}$$

Multiply the divisor to cancel the first term in the dividend when subtracted, then operate as in normal long division.

$$
\begin{array}{r}
x^2 \\
x-2\,\overline{)x^3 + 2 \cdot x^2 + 0 \cdot x + 12} \\
\underline{-x^3 + 2 \cdot x^2} \\
4 \cdot x^2 + 0 \cdot x + 12
\end{array}
$$

Repeat the process, which will result in $Q(x) = x^2 + 4x - 8$, and $R(x) = -4$. Verify with the division equality.

$$x^3 + 2 \cdot x^2 + 0 \cdot x + 12 = (x-2)(x^2 + 4x - 8) - 4$$

Partial Fractions Solves

Partial Fraction Decomposition

Given a rational function $N(x)/D(x)$, “expand” by the smallest possible roots of the denominator.

1) Linear roots are treated as follows:

$$\frac{N(x)}{(x+a)(x+b)} = \frac{A}{x+a} + \frac{B}{x+b}$$

2) Quadratic roots are treated as follows:

$$\frac{N(x)}{a \cdot x^2 + b \cdot x + c} = \frac{A \cdot x + B}{a \cdot x^2 + b \cdot x + c}$$

3) Repeated roots are treated as follows:

$$\frac{N(x)}{D(x)^p} = \frac{A}{D(x)} + \frac{B}{D(x)^2} + \cdots + \frac{C}{D(x)^p}$$

Example

$$\frac{5 \cdot x}{(x-1)(x^2+2)(x+7)^2} = \frac{A}{x-1} + \frac{B}{x+7} + \frac{C}{(x+7)^2} + \frac{D \cdot x + E}{x^2 + 2}$$

Solving by Example

Factor, then decompose the partial fraction

$$\frac{12 \cdot x}{x^2 - 10 \cdot x + 16} = \frac{12 \cdot x}{(x-8)(x-2)} = \frac{A}{x-8} + \frac{B}{x-2}$$

Multiply the equation by the denominator of the original rational function.

$$12 \cdot x = (x-2) \cdot A + (x-8) \cdot B$$

Enter values for x that would equate each term to zero, solving for the other term

$$x = 2 \therefore 12 \cdot 2 = 0 \cdot A - 6 \cdot B \therefore B = -4$$

$$x = 8 \therefore 12 \cdot 8 = 6 \cdot A + 0 \cdot B \therefore A = 16$$

Substitute A and B for their values to complete the decomposed partial fraction.

Logarithms

Definition

Logarithms are the inverse operation of exponentiation. The equality $b^p = x$ can be rewritten as $\log_b x = p$.

Example

$$2^3 = 8 \therefore \log_2 8 = 3$$

Base Inverse Property

$$\log_b b^p = p$$

Jump to Power Rule
Jump to Product and Quotient Rules
Jump to Cologarithms

Jump to Complex Number System
Jump to Analytic Trig Inverses

Proof

The value a logarithm uses to find an exponent is defined as the base to said exponential power.

$$\log_b x = p \wedge x = b^p \therefore \log_b b^p = p$$

Example

$$\log_2 8 = 3 \wedge 8 = 2^3 \therefore \log_2 2^3 = 3$$

Antilogarithms of an Equation

Given an equality, an equation may be raised as exponential powers with the same base value.

$$a = b \leftrightarrow x^a = x^b$$

Jump to L'Hôpital's Rule

Proof

Given $x^a = x^b$, take the logarithm of base x and apply the base inverse property for each side

Power Inverse Property

$$b^{\log_b x^p} = x^p, \forall x \in \mathbb{R}$$

Deductive Logic

Logarithms are the inverse of exponents, so operating a logarithm in an exponent with the same base cancels both.

Example

$$\log_2 2^3 = 2^{\log_2 3} = 3$$

Power Rule

$$\log_b x^p = p \cdot \log_b x$$

Jump to Change of Base Rule
Jump to Cologarithms
Jump to Derivative of Exponential Functions
Jump to Derivative of a Variable Raised to Itself

Proof

Let $a = \log_b x$, rewrite in the exponential form $x = b^a$, and then exponentiate to p.

$$x^p = (b^a)^p$$

Distribute the power, then take the logarithm using base b, and use the base inverse on the right.

$$\log_b x^p = a \cdot p$$

Substitute a for its original form.

Example

$$\log_2 4^8 = 8 \cdot \log_2 4 = 8 \cdot \log_2 2^2 = 2 \cdot 8 \cdot \cancel{\log_2 2} = 16$$

Product and Quotient Rules

$$\log_b(M \cdot N^{\pm 1}) = \log_b M \pm \log_b N$$

In the case that $M \cdot N^{\pm 1} = 1/N$, it follows that $\log_b(N^{-1}) = -\log_b N$

Jump to Sum and Difference Rules
Jump to Cologarithms
Jump to Complex Number System

Proof

Substitute the added terms with variables x and y, and write in exponential form.

$$x = \log_b M \land y = \log_b N \therefore M = b^x \land N = b^y$$

Multiply M and $N^{\pm 1}$, use the power rule for the exponents, and take the logarithm of both sides.

$$M \cdot N^{\pm 1} = b^{x \pm y} \therefore \log_b(M \cdot N^{\pm 1}) = \log_b b^{x \pm y}$$

Use the base inverse property to simplify the right, and substitute *x* and *y* for their original terms.

$$\log_b(M \cdot N^{\pm 1}) = x \pm y = \log_b M \pm \log_b N$$

Sum and Difference Rules

$$\log_b(M \pm N) = \log_b M + \log_b \left(1 \pm \frac{N}{M}\right)$$

Proof

Factor in M over itself as a multiple of *N*, then factor *M* out of the numerators, and use the product rule.

$$\log_b(M \pm N) = \log_b \left(M \pm \frac{M}{M} \cdot N\right) = \log_b \left(M \cdot \left(1 \pm \frac{N}{M}\right)\right) = \log_b M + \log_b \left(1 \pm \frac{N}{M}\right)$$

Change of Base Rule

$$\log_b x = \frac{\log_a x}{\log_a b}, x > 0 \land a > 0 \land a \neq 1$$

Jump to Derivative of Logarithmic Functions

Proof

Let $c = \log_b x$ so that $b^c = x$, then take the logarithm of base a on both sides.

$$\log_a b^c = \log_a x$$

Use the power rule to simplify the left side.

$$c \cdot \log_a b = \log_a x$$

Solve for *c* and substitute c for its original form.

Properties from the Change of Base Rule

1) $\log_b x = \dfrac{1}{\log_x b}$

2) $\log_{c^n} x = \dfrac{\log_c x}{n}$

3) $a^{\log_b x} = x^{\log_b a}$

Proofs of Properties

1) Using the change of base formula, substitute *x* for *a*.

2) Using the change of base formula, substitute c^n for *b* and *c* for *a*.

3) Take the logarithm *b* of the given formula and use power rule to obtain $\log_b x \cdot \log_b a = \log_b a \cdot \log_b x$.

Cologarithms

$$\log_{1/b} x = -\log_b x = \log_b x^{-1}$$

Proof

Let $a = \log_{1/b} x$ so that $x = (1/b)^a$, rewrite for power $-a$, and take logarithm *b* of x and b^{-a}.

$$a = \log_{1/b} x \therefore x = \left(\frac{1}{b}\right)^a = b^{-a} \longrightarrow \log_b x = \log_b b^{-a}$$

Apply the base inverse property on the right, negate, and substitute for a.

$$-(\log_b x) = -(-a) \longrightarrow -\log_b x = \log_{1/b} x$$

The equality $-\log_b x = \log_b(1/x)$ can be proven with the power rule and the quotient rule.

Basic & Analytic 2D Geometry

Lines

Slope of a Line

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Jump to Derivative Definition and Function
Jump to Mean Value Theorem for Derivatives

Horizontal and Vertical Lines

A horizontal line is the line *y* equal to a constant for all *x*, having a defined slope of zero.

A vertical line is the line *x* equal to a constant for all *y*, having an undefined slope of $\pm\infty$.

Slope-Intercept Equation

$$y(x) = m \cdot x + y_0$$

Point-Slope Equation

Given a line with point (x_1, y_1) and slope m,

$$y - y_1 = m \cdot (x - x_1)$$

Jump to Tangent Line

Two Point Equation

Given the point-slope equation with the slope formula substituted

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} \cdot (x - x_1)$$

Two-Intercept Equation

$$\frac{x}{x_0} + \frac{y}{y_0} = 1, x_0 \neq 0 \land y_0 \neq 0$$

Proof

Use the coordinates for the intercepts in the slope formula, which can be done interchangeably

$$m = \frac{0 - y_0}{x_0 - 0} = \frac{y_0 - 0}{0 - x_0} = -\frac{y_0}{x_0}$$

Substitute it into the slope-intercept formula and solve

$$y = y_0 - \frac{y_0}{x_0} \cdot x \therefore y + \frac{y_0}{x_0} \cdot x = y_0 \therefore \frac{y}{y_0} + \frac{\mathrm{x}}{x_0} = 1$$

General Equation

$$A \cdot x + B \cdot y = C, x_0 = \frac{C}{A} \land y_0 = \frac{C}{B}$$

Midpoint Formula

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

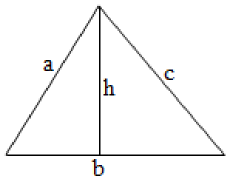
Two Line Relations

- Two lines are parallel if they have the same slope, i.e. $m_1 = m_2$
- Two lines intercept at (x, y), which is found by setting them equal to each other and yielding $x = (y_2 - y_1)/(m_1 - m_2)$
- Two lines are perpendicular if the slope of one is the negative reciprocal to the slope of the other, i.e. $m_1 = -1/m_2$

Triangles

Universal Properties

Closed planar objects with three straight sides, and a sum of lengths of any two sides greater than the third.

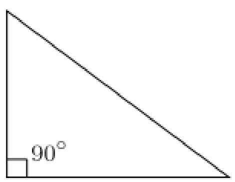


Perimeter	Area (non-obtuse)	Sum of angles
$a + b + c$	$\frac{1}{2} \cdot b \cdot h$	π

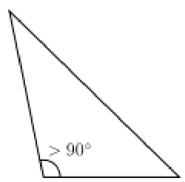
Jump to Right Angle Theorem
Jump to Universal Properties of Circles
Jump to Law of Sines

Types and Terminology

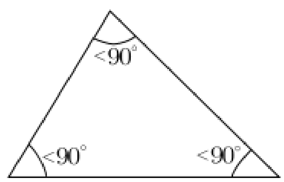
Right (all others are oblique)



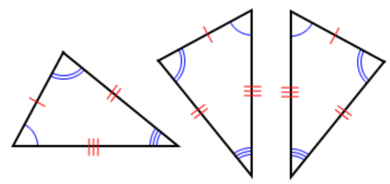
Obtuse



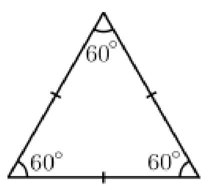
Acute



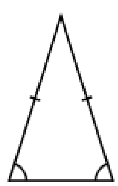
Congruent



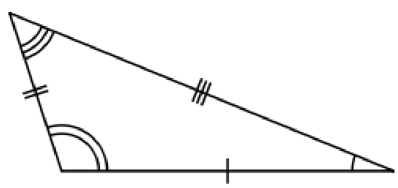
Equilateral



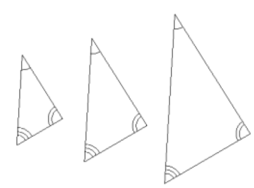
Isosceles



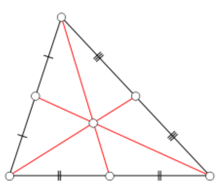
Scalene



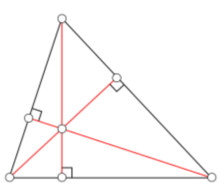
Similar



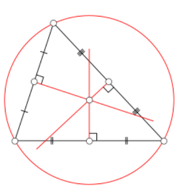
Intersections



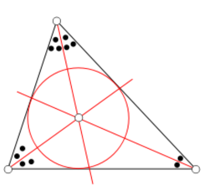
Centroid (G)
Median intersection point
Line segments extend from vertices (corners) to the sides at the midpoint



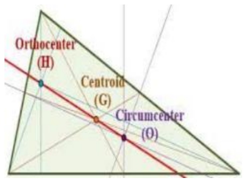
Orthocenter (H)
Altitude intersection point
Line segments extend from vertices to the sides at which right angles are formed



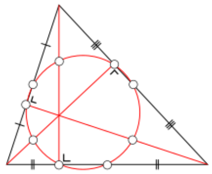
Circumcenter (O)
Perpendicular bisector intersection point
Line segments extend at right angles from the midpoint of each side



Incenter (I)
Angle bisector intersection point
Line segments extend from vertices dividing the angles of the vertices in half



Euler Line
Line passing through G, H and O of all non-equilateral triangles
(In equilateral triangles, G, H, O and I all overlap in a single point)

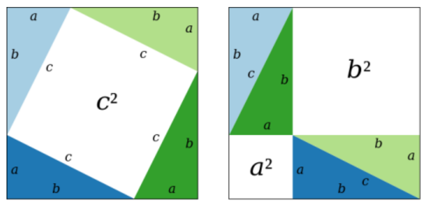


Nine-Point Circle
Circle passing though the midpoint of each side, the point of altitude on each side, and the midpoint of lines between each vertex and the orthocenter

Right Angle Theorem

$$h^2 = x^2 + y^2$$

Proof by rearrangement and algebra



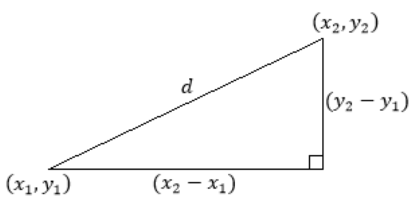
The square c^2 has area equal to all the triangle areas subtracted from the largest square.

$$c \cdot c = (a+b)(a+b) - 4 \cdot \left(\frac{1}{2} \cdot a \cdot b\right)$$

Expand all terms and cancel like terms.

$$c^2 = a^2 + 2 \cdot a \cdot b + b^2 - 2 \cdot a \cdot b = a^2 + b^2$$

Distance Formula of a Line



$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

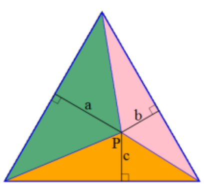
Jump to Trig Sum and Difference Identities

Proof

Using right angle theorem for (x_1, y_1) and (x_2, y_2). The magnitude is always a positive real value.

Vivaldi's Theorem

In an equilateral triangle, the sum of distances from any inner point to the sides is equal to the height of the triangle



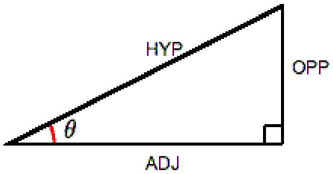
$$h = a + b + c$$

Proof

Given side s and height h, the area of the triangle equals to the areas of the inner triangles with altitudes a, b and c. Simplify.

$$\frac{s \cdot h}{2} = \frac{s \cdot a}{2} + \frac{s \cdot b}{2} + \frac{s \cdot c}{2} \therefore h = a + b + c$$

Trigonometric Right Angle Definitions



$$\sin(\theta) = \frac{opp}{hyp} \qquad \cos(\theta) = \frac{adj}{hyp} \qquad \tan(\theta) = \frac{opp}{adj}$$

$$\csc(\theta) = \frac{hyp}{opp} \qquad \sec(\theta) = \frac{hyp}{adj} \qquad \cot(\theta) = \frac{adj}{opp}$$

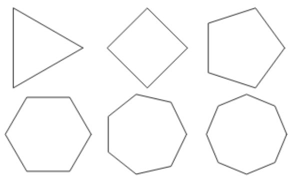
In solving for sides or angles of triangles, at least three elements must be known (SSS, SAS, ASA, AAS)

Jump to Regular Polygons
Jump to Universal Properties of Circles
Jump to Law of Sines

Polygons

Universal Properties

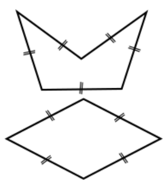
Closed planar objects with at least three straight sides and equal number of vertices (corners)



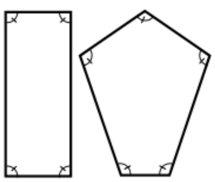
Perimeter	Area (non-obtuse)	Sum of angles
Sum of side lengths	Sum of areas of triangles divided within shape	$(\text{sides} - 2) \cdot \pi$

Types

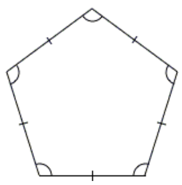
Equilateral



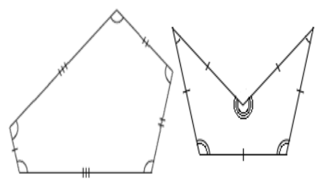
Equiangular



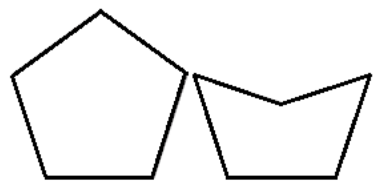
Regular



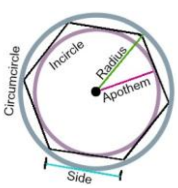
Irregular



Convex/Concave



Regular Polygons



Perimeter	Area	Sum of angles	Angle (h, r)	Circumradius
$n \cdot s$	$\frac{n \cdot s \cdot h}{2}$	$(n-2) \cdot \pi$	$\frac{\pi}{n}$	$h \cdot \sec\left(\frac{\pi}{n}\right), \frac{s}{2} \cdot \csc\left(\frac{\pi}{n}\right)$

Area using Apothem	Area using Side	Area using Circumradius
$A = n \cdot h^2 \cdot \tan\left(\frac{\pi}{n}\right)$	$A = \frac{n \cdot s^2}{4} \cdot \cot\left(\frac{\pi}{n}\right)$	$A = n \cdot r^2 \cdot \sin\left(\frac{\pi}{n}\right) \cdot \cos\left(\frac{\pi}{n}\right)$

- A = area
- n = number of sides
- s = length of sides
- h = apothem (incircle radius)
- r = circumradius

Jump to Universal Properties of Circles
Jump to Proof of π

Proof of Circumradius

Use the corresponding trig functions for h = adjacent, $b/2$ = opposite, and r = hypotenuse, and solve

$$\sec\left(\frac{\pi}{n}\right) = \frac{r}{h} \qquad \csc\left(\frac{\pi}{n}\right) = \frac{r}{s/2}$$

Proof of Area using Apothem

Set the two circumradius formulae equal to each other, multiply by h, and divide by the angle's cosecant to isolate $s \cdot h/2$

$$\frac{s \cdot h}{2} = h^2 \cdot \frac{\sec(\pi/n)}{\csc(\pi/n)}$$

Substitute the triangular elements of the secant and cosecant functions to simplify into one function, then substitute it

$$\frac{\sec(\pi/n)}{\csc(\pi/n)} = \frac{r/h}{r/s} = \frac{s}{h} = \tan\left(\frac{\pi}{n}\right) \therefore \frac{s \cdot h}{2} = h^2 \cdot \tan\left(\frac{\pi}{n}\right)$$

Substitute the function for $s \cdot h/2$ in the area formula

Proof of Area using Circumradius

Isolate h in the first circumradius equality, and substitute it into the area using the apothem equation

$$h = \frac{r}{\sec(\pi/n)} \rightarrow A = n \cdot r^2 \cdot \frac{\tan(\pi/n)}{\sec(\pi/n)^2}$$

Substitute the trig definitions of the tangent and secant functions to simplify to two functions, then substitute them

$$\frac{\tan(\pi/n)}{\sec(\pi/n)^2} = \frac{s/h}{r^2/h^2} = \frac{s \cdot h}{r \cdot r} = \sin\left(\frac{\pi}{n}\right) \cdot \cos\left(\frac{\pi}{n}\right) \therefore A = n \cdot r^2 \cdot \sin\left(\frac{\pi}{n}\right) \cdot \cos\left(\frac{\pi}{n}\right)$$

Proof of Area using Side Length

Substitute the second circumradius equality into the area using the circumradius equation, then expand

$$A = n \cdot \left(\frac{s}{2} \cdot \csc\left(\frac{\pi}{n}\right)\right)^2 \cdot \sin\left(\frac{\pi}{n}\right) \cdot \cos\left(\frac{\pi}{n}\right) = \frac{n \cdot s^2}{4} \cdot \csc\left(\frac{\pi}{n}\right)^2 \cdot \sin\left(\frac{\pi}{n}\right) \cdot \cos\left(\frac{\pi}{n}\right)$$

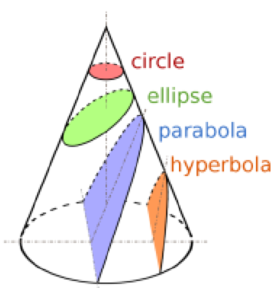
Substitute the trig definitions to simplify into one function, then substitute it

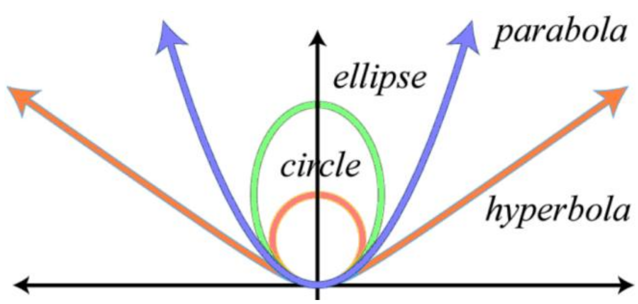
$$\csc\left(\frac{\pi}{n}\right)^2 \cdot \sin\left(\frac{\pi}{n}\right) \cdot \cos\left(\frac{\pi}{n}\right) = \frac{r^2}{s^2} \cdot \frac{s}{r} \cdot \frac{h}{r} = \frac{h}{s} = \cot\left(\frac{\pi}{n}\right) \therefore A = \frac{n \cdot s^2}{4} \cdot \cot\left(\frac{\pi}{n}\right)$$

Conic Sections

Universal Definitions

Two-dimensional subsets of a 3D (double) cone surface in which the shapes are determined by the intersection of a plane





General Equation

$$A \cdot x^2 + B \cdot x \cdot y + C \cdot y^2 + D \cdot x + E \cdot y + F = 0$$

Note: All conic sections will be without rotation in this section ($B = 0$). It will be presented in a future section.

Jump to Universal Properties of Circles
Jump to Universal Properties of Ellipses
Jump to Universal Properties of Parabolas
Jump to Universal Properties of Hyperbolas

Discriminant and Other Properties

$$\Delta = B^2 - 4 \cdot A \cdot C$$

Assuming no other values:

- If $B = 0$, then the function is not rotated
- If $\Delta < 0$, the function is an ellipse

- If $\Delta < 0$, $B = 0$ and $A < C$, the function is an ellipse with a horizontal major axis
- If $\Delta < 0$, $B = 0$ and $A > C$, the function is an ellipse with a vertical major axis
- If $\Delta < 0$, $B = 0$ and $A = C$, then the function is a circle
- If $\Delta < 0$, $A = C$, and $B = D = E = 0$, then the function is a circle centered at the origin
- If $\Delta = 0$, the function is a parabola
- If $C = D = F = 0$ then the function is a vertical parabola with the vertex at the origin
- If $A = E = F = 0$ then the function is a horizontal parabola with the vertex at the origin
- If $D = E = 0$, then the function is either an ellipse or hyperbola centered at the origin
- If $\Delta > 0$, the function is a hyperbola
- If $\Delta > 0$ and $A + C = 0$, the function is a rectangular hyperbola, meaning the asymptotes are perpendicular

Eccentricity

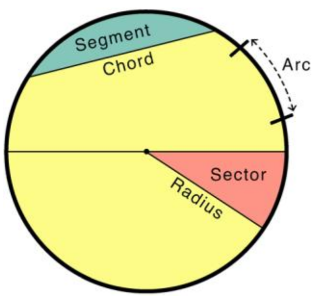
Eccentricity is a function's deviation from being circular

- Two conic sections are identical in shape if the eccentricity of each are equal
- Circle eccentricities are always 0
- Ellipse eccentricities are always between 0 and 1
- Parabola eccentricities are always 1
- Hyperbola eccentricities are always greater than 1
- Line eccentricities are infinite

Circles

Universal Properties

A closed curve with all points equidistant to an internal point



Circumference	Area	Arc Length	Sector Area	Chord Length (k)	Segment Area
$2 \cdot \pi \cdot r$	$\pi \cdot r^2$	$\theta \cdot r$	$\frac{\theta \cdot r^2}{2}$	$2 \cdot r \cdot \sin\left(\frac{\theta}{2}\right)$	$\frac{r^2}{2} \cdot \left(\theta - \sin\left(\frac{\theta}{2}\right) \cdot \cos\left(\frac{\theta}{2}\right)\right)$

Conic General Equation	$A \cdot (x^2 + y^2) + D \cdot x + E \cdot y + F = 0, A \neq 0$
Standard Equation	$(x - x_0)^2 + (y - y_0)^2 = r^2$
	(x_0, y_0) are the center coordinates
Conic-Standard Conversion	$x_0 = -\frac{D}{2 \cdot A}$ $\quad y_0 = -\frac{E}{2 \cdot A}$ $\quad r^2 = \frac{D^2 + E^2 - 4 \cdot A \cdot F}{4 \cdot A^2}$
Focus Coordinates	(x_0, y_0)
Eccentricity	0
Directrix	None

Deductive Logic for Area

Apply the area function of regular polygons, using the circumference as the perimeter and the radius as the apothem

$$A = (2 \cdot \pi \cdot r) \cdot \frac{r}{2}$$

Proof of Arc Length

The arc length is a fraction of the circumference, therefore can be found by the proportion to it and its angle

$$\frac{a}{2 \cdot \pi \cdot r} = \frac{\theta}{2 \cdot \pi}$$

Proof of Sector Area

The sector area is a fraction of the circle area, therefore can be found by the proportion to it and its angle

$$\frac{A_s}{\pi \cdot r^2} = \frac{\theta}{2 \cdot \pi}$$

Proof of Chord Length

Use the radius, half angle, and half the chord length to form a right triangle, then use the sine function and solve

$$\sin\left(\frac{\theta}{2}\right) = \frac{k}{2 \cdot r}$$

Proof of Segment Area

The segment area is the triangular area between the center and chord subtracted from the sector area

$$A_s = \frac{\theta \cdot r^2}{2} - A_t$$

Substitute the triangular area using trig definitions solving for $b = \mathrm{r} \cdot \sin(\theta/2)$ and $h = \mathrm{r} \cdot \cos(\theta/2)$, and simplify

$$A_s = \frac{\theta \cdot r^2}{2} - \frac{r^2}{2} \cdot \sin\left(\frac{\theta}{2}\right) \cdot \cos\left(\frac{\theta}{2}\right) = \frac{r^2}{2} \cdot \left(\theta - \sin\left(\frac{\theta}{2}\right) \cdot \cos\frac{\theta}{2}\right)$$

Conic-Standard Conversion

Given the conic general equation with the properties for a circle, group the x terms and y terms, and isolate the constant

$$A \cdot x^2 + A \cdot y^2 + D \cdot x + E \cdot y + F = 0 \therefore A \cdot x^2 + D \cdot x + A \cdot y^2 + E \cdot y = -F$$

Divide by A, and complete the square for the x terms and y terms

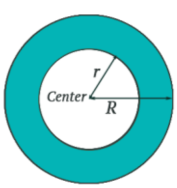
$$x^2 + \frac{D}{A} \cdot x + \frac{D^2}{4 \cdot A^2} + y^2 + \frac{E}{A} \cdot y + \frac{E^2}{4 \cdot A^2} = -\frac{F}{A} + \frac{D^2}{4 \cdot A^2} + \frac{E^2}{4 \cdot A^2}$$

Factor. The equation is in standard form.

$$\left(x + \frac{D}{2 \cdot A}\right)^2 + \left(y + \frac{E}{2 \cdot A}\right)^2 = \frac{D^2 + E^2 - 4 \cdot A \cdot F}{4 \cdot A^2}$$

Annulus

A ring formed by two concentric circles. All basic features of circles apply, with respect to differences involving two radii.

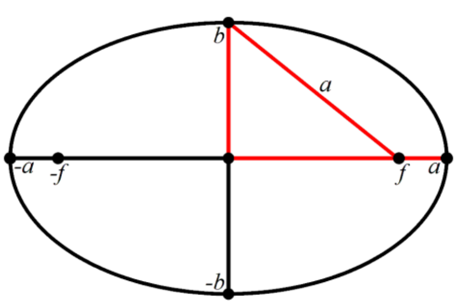


Perimeter	Area	Sector Area
$2 \cdot \pi \cdot (R + r)$	$\pi \cdot (R^2 - r^2)$	$\frac{\theta \cdot (R^2 - r^2)}{2}$

Ellipses

Universal Properties

A closed ovular curve whose points are the sum of the distances from two internal points



Perimeter	Area
$4 \cdot a < P < 2 \cdot \pi \cdot a, a > b$	$\pi \cdot a \cdot b$

Conic General Equation	$A \cdot x^2 + C \cdot y^2 + D \cdot x + E \cdot y + F = 0, A \cdot C > 0$
Standard Equation	$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1$ (x_0, y_0) are the center coordinates
Conic-Standard Conversion	$a^2 = C$ $x_0 = -\frac{D}{2 \cdot A}$ $b^2 = A$ $y_0 = -\frac{E}{2 \cdot C}$
Standard-Conic Conversion	$A = b^2$ $D = -2 \cdot b^2 \cdot x_0$ $C = a^2$ $E = -2 \cdot a^2 \cdot y_0$ $F = b^2 \cdot x_0^2 + a^2 \cdot y_0^2 - a^2 \cdot b^2$
Orientation	$\left.\begin{matrix} C > A \\ a > b \end{matrix}\right\}$ elongated horizontally
Foci Coordinates	$\left(x_0 \pm \sqrt{a^2 - b^2}, y_0\right), a > b \vee \left(x_0, y_0 \pm \sqrt{b^2 - a^2}\right), b > a$
Eccentricity	$e = \frac{f}{a} = \sqrt{1 - \frac{b^2}{a^2}}, a > b \vee e = \frac{f}{b} = \sqrt{1 - \frac{a^2}{b^2}}, b > a, e \in (0,1)$
Directrix	$x = \pm \frac{a}{e}, a > b \vee y = \pm \frac{b}{e}, b > a$

Standard-Conic Conversion

Jump to Universal Properties of Hyperbolas

Given the standard equation, multiply by $a^2 \cdot b^2$ and expand

$$b^2 \cdot x^2 - 2 \cdot b^2 \cdot x \cdot x_0 + b^2 \cdot x_0^2 + a^2 \cdot y^2 - 2 \cdot a^2 \cdot y \cdot y_0 + a^2 \cdot y_0^2 = a^2 \cdot b^2$$

Rearrange to appear as the conic general equation, and use the coefficient method of solving

$$b^2 \cdot x^2 + a^2 \cdot y^2 - 2 \cdot b^2 \cdot x_0 \cdot x - 2 \cdot a^2 \cdot y_0 \cdot y + b^2 \cdot x_0^2 + a^2 \cdot y_0^2 - a^2 \cdot b^2 = 0$$

Conic-Standard Conversion

Jump to Universal Properties of Hyperbolas

Given the conic general equation, group the x terms and y terms, and isolate the constant

$$A \cdot x^2 + C \cdot y^2 + D \cdot x + E \cdot y + F = 0 \therefore A \cdot x^2 + D \cdot x + C \cdot y^2 + E \cdot y = -F$$

Factor with x^2 and y^2 having coefficients of 1, and complete the square for the x terms and y terms

$$A \cdot \left(x^2 + \frac{D}{A} \cdot x + \frac{D^2}{4 \cdot A^2}\right) + C \cdot \left(y^2 + \frac{E}{C} \cdot y + \frac{E^2}{4 \cdot C^2}\right) = -F + A \cdot \left(\frac{D^2}{4 \cdot A^2}\right) + C \cdot \left(\frac{E^2}{4 \cdot C^2}\right)$$

Factor

$$A \cdot \left(x + \frac{D}{2 \cdot A}\right)^2 + C \cdot \left(y + \frac{E}{2 \cdot C}\right)^2 = \frac{C \cdot D^2 + A \cdot (E^2 - 4 \cdot C \cdot F)}{4 \cdot A \cdot C}$$

Divide by the constant. The equation is in standard form.

$$\left(x + \frac{D}{2 \cdot A}\right)^2 \Big/ \left(\frac{C \cdot D^2 + A \cdot (E^2 - 4 \cdot C \cdot F)}{4 \cdot A^2 \cdot C}\right) + \left(y + \frac{E}{2 \cdot C}\right)^2 \Big/ \left(\frac{C \cdot D^2 + A \cdot (E^2 - 4 \cdot C \cdot F)}{4 \cdot A \cdot C^2}\right) = 1$$

Set the denominators equal to their conic equivalents, and substitute their factors with their standard-conic conversions

$$C = \frac{C \cdot D^2 + A \cdot (E^2 - 4 \cdot C \cdot F)}{4 \cdot A^2 \cdot C} = \frac{a^2 \cdot (-2 \cdot b^2 \cdot x_0)^2 + b^2 \cdot \left((-2 \cdot a^2 \cdot y_0)^2 - 4 \cdot a^2 \cdot (b^2 \cdot x_0^2 + a^2 \cdot y_0^2 - a^2 \cdot b^2)\right)}{4 \cdot (b^2)^2 \cdot a^2}$$

$$A = \frac{C \cdot D^2 + A \cdot (E^2 - 4 \cdot C \cdot F)}{4 \cdot A \cdot C^2} = \frac{a^2 \cdot (-2 \cdot b^2 \cdot x_0)^2 + b^2 \cdot \left((-2 \cdot a^2 \cdot y_0)^2 - 4 \cdot a^2 \cdot (b^2 \cdot x_0^2 + a^2 \cdot y_0^2 - a^2 \cdot b^2)\right)}{4 \cdot b^2 \cdot (a^2)^2}$$

Expand, cancel like terms, and simplify

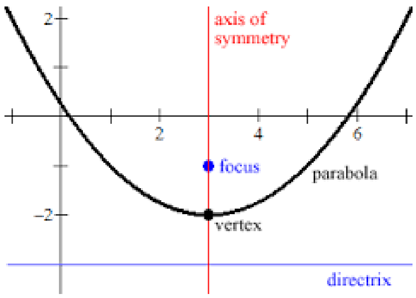
$$C = \frac{4 \cdot a^2 \cdot b^4 \cdot x_0^2 + 4 \cdot a^4 \cdot b^2 \cdot y_0^2 - 4 \cdot a^2 \cdot b^4 \cdot x_0^2 - 4 \cdot a^4 \cdot b^2 \cdot y_0^2 + 4 \cdot a^4 \cdot b^4}{4 \cdot b^4 \cdot a^2} = a^2$$

$$A = \frac{4 \cdot a^2 \cdot b^4 \cdot x_0^2 + 4 \cdot a^4 \cdot b^2 \cdot y_0^2 - 4 \cdot a^2 \cdot b^4 \cdot x_0^2 - 4 \cdot a^4 \cdot b^2 \cdot y_0^2 + 4 \cdot a^4 \cdot b^4}{4 \cdot b^2 \cdot a^4} = b^2$$

Parabolas

Universal Properties

An open mirrored curve whose points are the same distance between a common internal point and an external line



Conic General Equations	Vertical: $A \cdot x^2 + D \cdot x + E \cdot y + F = 0$
	Horizontal: $C \cdot y^2 + D \cdot x + E \cdot y + F = 0$
Standard Equation	$y = a \cdot x^2 + b \cdot x + y_0$
Vertex Form	$y = a \cdot (x - x_0)^2 + y_0$
	(x_0, y_0) are the vertex coordinates
Intercept Form	$y = a \cdot (x - x_1)(x - x_2)$
Discriminant	$\Delta = b^2 - 4 \cdot a \cdot y_0$

- If Δ<0, two x-intercepts
- If Δ=0, one x-intercept
- If Δ>0, no x-intercepts

Conic-Standard Conversions	$a = -A/E$	$b - D/E$	$y_0 = -F/E$

Vertex Coordinates	$(x_0, y_0) = \left(-\frac{b}{2 \cdot a}, y_0 - \frac{b^2}{4 \cdot a}\right)$
x Intercepts	$\{(x_1, 0), (x_2, 0)\} = \left(\frac{-b \pm \sqrt{b^2 - 4 \cdot a \cdot y_0}}{2 \cdot a}, 0\right)$
Focus Length from Vertex	$f = 1/(4 \cdot a)$
Focus Coordinates	$\left(-\frac{b}{2 \cdot a}, y_0 - \frac{b^2 + 1}{4 \cdot a}\right)$
Eccentricity	1
Directrix	$y = -f$

Conic-Standard Conversion

Rearrange the conic general equation to isolate the y term, and divide by E. The equation is in standard form.

$$A \cdot x^2 + D \cdot x + E \cdot y + F = 0 \therefore y = -\frac{A}{E} \cdot x^2 - \frac{D}{E} \cdot x - \frac{F}{E}$$

Standard-Vertex Conversion

Given the standard equation, isolate the x terms, divide by a, and complete the square

$$\frac{y-y_0}{a}+\frac{b^2}{4\cdot a^2}=x^2+\frac{b}{a}\cdot x+\frac{b^2}{4\cdot a^2}=\left(x+\frac{b}{2\cdot a}\right)^2$$

Isolate y. The equation is in vertex form.

$$y=a\cdot\left(x+\frac{b}{2\cdot a}\right)^2-\frac{b^2}{4\cdot a}+y_0$$

Standard-Intercept Conversion

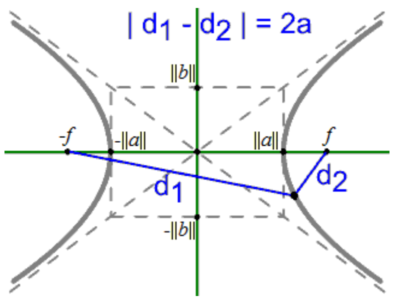
Given the quadratic formula with $y=0$, find the zeros of x.

$$x=\frac{-b\pm\sqrt{b^2-4\cdot a\cdot y_0}}{2\cdot a}$$

Hyperbolas

Universal Properties

A mirrored set of open mirrored curves whose points are the difference between two common internal points



Conic General Equation	$A\cdot x^2+C\cdot y^2+D\cdot x+E\cdot y+F=0, A\cdot C<0$
Standard Equation	$\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}=1, \mathrm{a}^2\cdot\mathrm{b}^2<0$ (x_0,y_0) are the center coordinates
Standard Equation (Real Terms Only)	$\pm\frac{(x-x_0)^2}{a^2}\mp\frac{(y-y_0)^2}{b^2}=1$
Conic-Standard Conversion	$a^2=C$ $x_0=-\frac{D}{2\cdot A}$ $b^2=A$ $y_0=-\frac{E}{2\cdot C}$
Standard-Conic Conversion	$A=b^2$ $D=-2\cdot b^2\cdot x_0$ $C=a^2$ $E=-2\cdot a^2\cdot y_0$ $F=b^2\cdot x_0^2+a^2\cdot y_0^2-a^2\cdot b^2$
Orientation	$C<0;\ a\in\mathbb{C}$ opens horizontally $A<0;\ b\in\mathbb{C}$ opens vertically
Vertices	$(x_0\pm a,y_0)\vee(x_0,y_0\pm b)$
Focus Coordinates	$\left(x_0\pm\sqrt{a^2+b^2},y_0\right)\vee\left(x_0,y_0\pm\sqrt{a^2+b^2}\right)$
Eccentricity	$e=\frac{f}{a}=\sqrt{1+\frac{b^2}{a^2}}$
Directrix	$x=\pm a^2/f\vee y=\pm b^2/f$
Asymptotes	$y=\pm\frac{b\cdot(x-x_0)}{a}+y_0$

Conversions

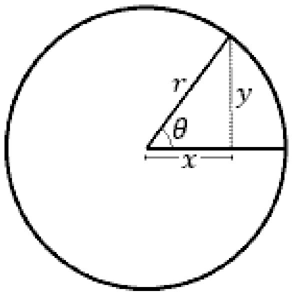
The proofs are the same for ellipses, however since by definition $A \cdot C < 0$, either a or b must be imaginary.

Trigonometry

Definitions and Basics

Trig Unit Circle

The trig functions are the functions of rotation on a circumference



$$\sin(\theta) = \frac{y}{r} \qquad \cos(\theta) = \frac{x}{r} \qquad \tan(\theta) = \frac{y}{x}$$

$$\csc(\theta) = \frac{r}{y} \qquad \sec(\theta) = \frac{r}{x} \qquad \cot(\theta) = \frac{x}{y}$$

$$\theta = \sin^{-1}\left(\frac{y}{r}\right) \qquad \theta = \cos^{-1}\left(\frac{x}{r}\right) \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\theta = \csc^{-1}\left(\frac{r}{y}\right) \qquad \theta = \sec^{-1}\left(\frac{r}{x}\right) \qquad \theta = \cot^{-1}\left(\frac{x}{y}\right)$$

Jump to Sum and Difference Identities

Inverse, Reciprocal and Power Notation

Sources reference inverses as $\arcsin(\theta)$ to avoid confusion. Here they are listed as above while reciprocal functions are as

$$\sin(\theta)^{-1} = \frac{1}{\sin(\theta)} \neq \sin^{-1}(\theta)$$

However,

$$\sin(\theta)^n = \sin^n(\theta)\ \forall n \neq -1$$

Even/Odd Identities and Reflections

$$\sin(-\theta) = -\sin(\theta) \qquad \cos(-\theta) = \cos(\theta) \qquad \tan(-\theta) = -\tan(\theta)$$

$$csc(-\theta) = -csc(\theta) \qquad sec(-\theta) = sec(\theta) \qquad cot(-\theta) = -cot(\theta)$$

The yielded sign value depends in which quadrant are the values of (x, y)

I: $0 \le \theta \le \pi/2$	**II: $\pi/2 < \theta \le \pi$**	**III: $\pi/2 < \theta < 3 \cdot \pi/2$**	**IV: $3 \cdot \pi/2 \le \theta < 2 \cdot \pi$**
$\sin(\theta): +$ $\csc(\theta): +$	$\sin(\theta): +$ $\csc(\theta): +$	$\sin(\theta): -$ $\csc(\theta): -$	$\sin(\theta): -$ $\csc(\theta): -$
$\cos(\theta): +$ $\sec(\theta): +$	$\cos(\theta): -$ $\sec(\theta): -$	$\cos(\theta): -$ $\sec(\theta): -$	$\cos(\theta): +$ $\sec(\theta): +$
$\tan(\theta): +$ $\cot(\theta): +$	$\tan(\theta): -$ $\cot(\theta): -$	$\tan(\theta): +$ $\cot(\theta): +$	$\tan(\theta): -$ $\cot(\theta): -$

Jump to Sum and Difference Identities

Cofunctions and Complementary Angle Identities

Cofunctions

$$\sin(\theta) = \frac{1}{\csc(\theta)} \qquad \cos(\theta) = \frac{1}{\sec(\theta)} \qquad \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{1}{\cot(\theta)}$$

Jump to Derivatives of Trig Functions

Complementary Angles

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta \qquad \csc\left(\frac{\pi}{2} - \theta\right) = \sec\theta \qquad \tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$$

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta \qquad \sec\left(\frac{\pi}{2} - \theta\right) = \csc\theta \qquad \cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta$$

Jump to Sum and Difference Identities

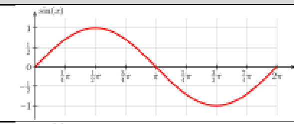
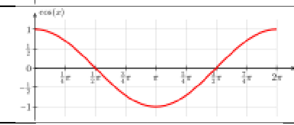
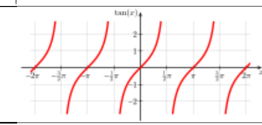
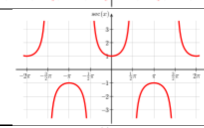
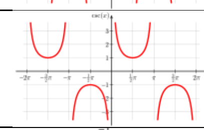
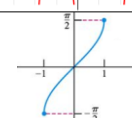
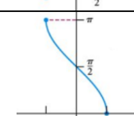
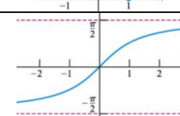
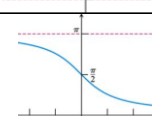
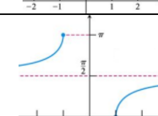
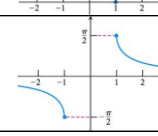
Inverse Cofunctions

$\sin^{-1}\left(\frac{1}{x}\right) = \csc^{-1}(x), |x| \geq 1$ $\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1}(x), |x| \geq 1$ $\tan^{-1}\left(\frac{1}{x}\right) = \cot^{-1}(x), x > 0$

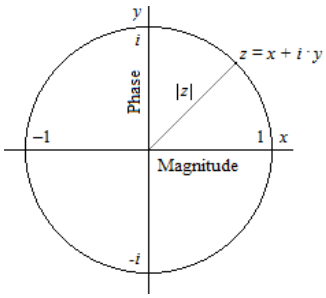
Inverse Complementary Angles

$\sin^{-1}(x) + \cos^{-1}(x) = \frac{\pi}{2}, |x| \leq 1$ $\tan^{-1}(x) + \cot^{-1}(x) = \frac{\pi}{2}$ $\csc^{-1}(x) + \sec^{-1}(x) = \frac{\pi}{2}, |x| \geq 1$

Graphs

Function	Period	Domain	Range	Graph
$\sin(\theta)$	$2 \cdot \pi$	$(-\infty, \infty)$	$[-1,1]$	
$\cos(\theta)$	$2 \cdot \pi$	$(-\infty, \infty)$	$[-1,1]$	
$\tan(\theta)$	π	$(-\infty, \infty), x \neq (2 \cdot n - 1) \cdot \frac{\pi}{2}\ \forall n \in \mathbb{Z}$	$(-\infty, \infty)$	
$\cot(\theta)$	π	$(-\infty, \infty), x \neq (2 \cdot n - 1) \cdot \frac{\pi}{2}\ \forall n \in \mathbb{Z}$	$(-\infty, \infty)$	
$\sec(\theta)$	$2 \cdot \pi$	$(-\infty, \infty), x \neq (2 \cdot n - 1) \cdot \frac{\pi}{2}\ \forall n \in \mathbb{Z}$	$|y| \geq 1$	
$\csc(\theta)$	$2 \cdot \pi$	$(-\infty, \infty), x \neq 2 \cdot n \cdot \pi\ \forall n \in \mathbb{Z}$	$|y| \geq 1$	
$\sin^{-1}(\theta)$	—	$|x| \leq 1$	$|y| \leq \frac{\pi}{2}$	
$\cos^{-1}(\theta)$	—	$|x| \leq 1$	$0 \leq y \leq \pi$	
$\tan^{-1}(\theta)$	—	$(-\infty, \infty)$	$|y| \leq \frac{\pi}{2}$	
$\cot^{-1}(\theta)$	—	$(-\infty, \infty)$	$0 < y < \pi$	
$\sec^{-1}(\theta)$	—	$|x| \geq 1$	$0 \leq y \leq \pi, y \neq \frac{\pi}{2}$	
$\csc^{-1}(\theta)$	—	$|x| \geq 1$	$|y| \leq \frac{\pi}{2}, y \neq 0$	

Complex Unit Circle



- The complex plain is a two-dimensional number set whose exponential nature is rotational
- Rational powers of i expand into complex numbers with multipliers of 1 and i^1
- Powers of i^x function the same as radians on the trig unit circle with $x = 2$ equivalent to one radian
- It follows that i^x is a two component number on the unit circle equal to $\cos(x \cdot \pi/2) + i \cdot \sin(x \cdot \pi/2)$

Jump to Complex Number System

Examples of Imaginary Units Exponentiated

$$\sqrt{i} = \cos\left(\frac{\pi}{4}\right) + i \cdot \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \cdot i$$

$$i^{5/3} = \cos\left(\frac{5 \cdot \pi}{6}\right) + i \cdot \sin\left(\frac{5 \cdot \pi}{6}\right) = -\frac{\sqrt{3}}{2} + \frac{1}{2} \cdot i$$

Common Identities

Right Angle Identities

$$\sin^2(\theta) + \cos^2(\theta) = 1 \qquad \tan^2(\theta) + 1 = \sec^2(\theta) \qquad \cot^2(\theta) + 1 = \csc^2(\theta)$$

Jump to Sum and Difference Identities
Jump to Half Angle Identities
Jump to Limits of Trig Functions
Jump to Derivatives of Trig Functions
Jump to Derivatives of Trig Inverses

Proof of Sine and Cosine

Given the right angle theorem, divide by the hypotenuse, and substitute trig right angle definitions

$$\frac{h^2 = x^2 + y^2}{h^2} \rightarrow 1 = \frac{x^2}{h^2} + \frac{y^2}{h^2} \rightarrow 1 = \cos^2(\theta) + \sin^2(\theta)$$

Proof of Tangent and Secant

Given the right angle theorem, divide by the x component, and substitute trig right angle definitions

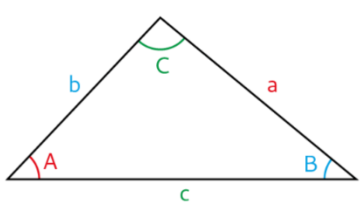
$$\frac{h^2 = x^2 + y^2}{x^2} \rightarrow \frac{h^2}{x^2} = 1 + \frac{y^2}{x^2} \rightarrow \sec^2(\theta) = 1 + \tan^2(\theta)$$

Proof of Cotangent and Cosecant

Given the right angle theorem, divide by the y component, and substitute trig right angle definitions

$$\frac{h^2 = x^2 + y^2}{y^2} \rightarrow \frac{h^2}{x^2} = \frac{x^2}{y^2} + 1 \rightarrow \csc^2(\theta) = \cot^2(\theta) + 1$$

Law of Sines



$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

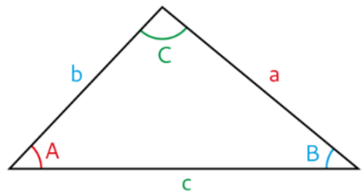
Proof

Given the triangle area, substitute the height of each side with its right angle definition for sine

$$\frac{1}{2} \cdot b \cdot c \cdot \sin(A) = \frac{1}{2} \cdot a \cdot c \cdot \sin(B) = \frac{1}{2} \cdot a \cdot b \cdot \sin(C)$$

Multiply by $2/(a \cdot b \cdot c)$

Law of Cosines



$$c^2 = a^2 + b^2 - 2 \cdot a \cdot b \cdot \cos(C) \qquad C = \cos^{-1}\left(\frac{a^2 + b^2 - c^2}{2 \cdot a \cdot b}\right)$$

Proof

Insert the perpendicular line to c to represent the height of the triangle, dividing c as

$$c = c_1 + c_2$$

Find c_1 and c_2 by taking the cosine of a and b, then solving

$$\cos(A) = \frac{c_2}{b} \wedge \cos(B) = \frac{c_1}{a} \therefore b \cdot \cos(A) = c_2 \wedge a \cdot \cos(B) = c_1$$

Substitute the values for c_1 and c_2 for the sum of c, then multiply by c. Repeat this process for the other two sides.

$$c^2 = b \cdot c \cdot \cos(A) + a \cdot c \cdot \cos(B) \qquad a^2 = a \cdot b \cdot \cos(C) + a \cdot c \cdot \cos(B) \qquad b^2 = a \cdot b \cdot \cos(C) + b \cdot c \cdot \cos(A)$$

Add the last two equations and subtract the first, simplify, and isolate c^2

$$a^2 + b^2 - c^2 = a \cdot b \cdot \cos(C) + a \cdot c \cdot \cos(B) + a \cdot b \cdot \cos(C) + b \cdot c \cdot \cos(A) - b \cdot c \cdot \cos(A) - a \cdot c \cdot \cos(B)$$

Repeat the process for the other two possible combinations, isolating a^2 and b^2 respectively

$$a^2 = b^2 + c^2 - 2 \cdot b \cdot c \cdot \cos(A) \qquad b^2 = a^2 + c^2 - 2 \cdot a \cdot c \cdot \cos(B)$$

Sum and Difference Identities

$$\sin(a \pm b) = \sin(a) \cdot \cos(b) \pm \cos(a) \cdot \sin(b) \qquad \cos(a \pm b) = \cos(a) \cdot \cos(b) \mp \sin(a) \cdot \sin(b)$$

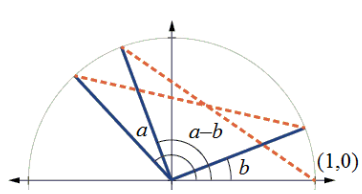
$$\csc(a \pm b) = \frac{1}{\sin(a) \cdot \cos(b) \pm \cos(a) \cdot \sin(b)} \qquad \sec(a \pm b) = \frac{1}{\cos(a) \cdot \cos(b) \mp \sin(a) \cdot \sin(b)}$$

$$\tan(a \pm b) = \frac{\tan(a) \pm \tan(b)}{1 \mp \tan(a) \cdot \tan(b)} \qquad \cot(a \pm b) = \frac{\cot(a) \cdot \cot(b) \mp 1}{\cot(b) \pm \cot(a)}$$

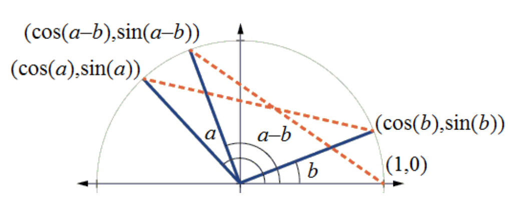
Jump to Double Angle Identities
Jump to Derivatives of Trig Functions

Proof of Sine and Cosine

On the unit circle, plot points for $(1,0)$, and for the angles b, $a - b$, and a, respectively in a positive rotation



Determine the coordinates of each point given their angles using the standard trig definitions



The distances displayed are equal to each other. Use the distance formula for each set equal to each other.

$$\sqrt{(1-\cos(a-b))^2+(0-\sin(a-b))^2}=\sqrt{(\cos(b)-\cos(a))^2+(\sin(b)-\sin(a))^2}$$

Square, expand the left, then use the right angle identity to simplify to one

$$1-2\cdot\cos(a-b)+(\cos^2(a-b)+\sin^2(a-b)=1)=(\cos(b)-\cos(a))^2+(\sin(b)-\sin(a))^2$$

Expand the right, then use the right angle identity to simplify to one twice

$$2-2\cdot\cos(a-b)=\cos^2(b)-2\cdot\cos(a)\cdot\cos(b)+\cos^2(a)+\sin^2(b)-2\cdot\sin(a)\cdot\sin(b)+\sin^2(a)+(2)$$

Subtract 2, then divide by negative 2

$$\cos(a-b)=\cos(a)\cdot\cos(b)+\sin(a)\cdot\sin(b)$$

For the cosine sum identity, use $-b$ as b, and apply the even/odd identities

$$\cos\big(a-(-b)\big)=\cos(a)\cdot\cos(-b)+\sin(a)\cdot\sin(-b)\ \rightarrow\ \cos(a+b)=\cos(a)\cdot\cos(b)-\sin(a)\cdot\sin(b)$$

For the sine difference identity use $\pi/2-a$ for a and $\pi/2-(a+b)$ for $a+b$ in the cosine sum identity

$$\cos\left(\frac{\pi}{2}-(a+b)\right)=\cos\left(\frac{\pi}{2}-a\right)\cdot\cos(b)-\sin\left(\frac{\pi}{2}-a\right)\cdot\sin(b)$$

Apply the complimentary angle identities

$$\sin(a-b)=\sin(a)\cdot\cos(b)-\cos(a)\cdot\sin(b)$$

For the sine sum identity, use $-b$ as b, and apply the even/odd identities

$$\sin\big(a-(-b)\big)=\sin(a)\cdot\cos(-b)-\cos(a)\cdot\sin(-b)\ \rightarrow\ \sin(a+b)=\sin(a)\cdot\cos(b)+\cos(a)\cdot\sin(b)$$

Proof of Tangent

Given that tangent can be represented as sine over cosine, use the sum and difference formulas for each

$$\tan(a\pm b)=\frac{\sin(a\pm b)}{\cos(a\pm b)}=\frac{\sin(a)\cdot\cos(b)\pm\cos(a)\cdot\sin(b)}{\cos(a)\cdot\cos(b)\mp\sin(a)\cdot\sin(b)}$$

Divide the numerator and denominator by $\cos(a)\cdot\cos(b)$, then cancel like terms in each

$$\tan(a\pm b)=\frac{\sin(a)\cdot\cos(b)\pm\cos(a)\cdot\sin(b)}{\cos(a)\cdot\cos(b)\mp\sin(a)\cdot\sin(b)}\cdot\frac{1/(\cos(a)\cdot\cos(b))}{1/(\cos(a)\cdot\cos(b))}=\frac{\sin(a)/\cos(a)\pm\sin(b)/\cos(b)}{1\mp(\sin(a)\cdot\sin(b))/(\cos(a)\cdot\cos(b))}$$

Substitute the tangent function for the sines over cosines

Proof of Cotangent

Given that cotangent can be represented as cosine over sine, use the sum and difference formulas for each

$$\cot(a\pm b)=\frac{\cos(a\pm b)}{\sin(a\pm b)}=\frac{\cos(a)\cdot\cos(b)\mp\sin(a)\cdot\sin(b)}{\sin(a)\cdot\cos(b)\pm\cos(a)\cdot\sin(b)}$$

Divide the numerator and denominator by $\sin(a)\cdot\sin(b)$, then cancel like terms in each

$$\cot(a\pm b)=\frac{\cos(a)\cdot\cos(b)\mp\sin(a)\cdot\sin(b)}{\sin(a)\cdot\cos(b)\pm\cos(a)\cdot\sin(b)}\cdot\frac{1/(\sin(a)\cdot\sin(b))}{1/(\sin(a)\cdot\sin(b))}=\frac{(\cos(a)\cdot\cos(b))/(\sin(a)\cdot\sin(b))\mp 1}{\cos(b)/\sin(b)\pm\cos(a)/\sin(a)}$$

Substitute the cotangent function for the cosines over sines

Product-to-Sum Identities

$$2\cdot\sin(a)\cdot\sin(b)=\cos(a-b)-\cos(a+b)\qquad 2\cdot\sin(a)\cdot\cos(b)=\sin(a+b)+\sin(a-b)$$

$$2\cdot\cos(a)\cdot\cos(b)=\cos(a-b)+\cos(a+b)\qquad 2\cdot\cos(a)\cdot\sin(b)=\sin(a+b)-\sin(a-b)$$

Proofs

Substitute the sum and difference identities for the terms on the right, cancel like terms, and factor

Sum-to-Product Identities

$$\sin(a) \pm \sin(b) = 2 \cdot \sin\left(\frac{a \pm b}{2}\right) \cdot \cos\left(\frac{a \mp b}{2}\right)$$

$$\cos(a) + \cos(b) = 2 \cdot \cos\left(\frac{a + b}{2}\right) \cdot \cos\left(\frac{a - b}{2}\right)$$

$$\cos(a) - \cos(b) = -2 \cdot \sin\left(\frac{a + b}{2}\right) \cdot \sin\left(\frac{a - b}{2}\right)$$

Proofs

Using the angles from the product-to-sum identities, let $a = (u + v)/2$ and $b = (u - v)/2$ such that

$$a + b = \frac{u + v}{2} + \frac{u - v}{2} = u \qquad a - b = \frac{u + v}{2} - \frac{u - v}{2} = v$$

Substitute the values for the product-to-sum identities, (and rename the variables).

Double Angle Identities

$$\sin(2 \cdot \theta) = 2 \cdot \sin(\theta) \cdot \cos(\theta) \qquad \cos(2 \cdot \theta) = \cos^2(\theta) - \sin^2(\theta) \qquad \tan(2 \cdot \theta) = \frac{2 \cdot \tan(\theta)}{1 - \tan^2(\theta)}$$

$$\csc(2 \cdot \theta) = \frac{1}{2} \cdot \sec(\theta) \cdot \csc(\theta) \qquad \sec(2 \cdot \theta) = \frac{1}{\cos^2(\theta) - \sin^2(\theta)} \qquad \cot(2 \cdot \theta) = \frac{\cot^2(\theta) - 1}{2 \cdot \cot(\theta)}$$

Proofs

Use the same angle twice in the sum identities for each function

Half Angle Identities

$$\left|\sin\left(\frac{\theta}{2}\right)\right| = \sqrt{\frac{1 - \cos(\theta)}{2}} \qquad \left|\cos\left(\frac{\theta}{2}\right)\right| = \sqrt{\frac{1 + \cos(\theta)}{2}} \qquad \left|\tan\left(\frac{\theta}{2}\right)\right| = \sqrt{\frac{1 - \cos(\theta)}{1 + \cos(\theta)}} \qquad \left|\cot\left(\frac{\theta}{2}\right)\right| = \sqrt{\frac{1 + \cos(\theta)}{1 - \cos(\theta)}}$$

Proof of Sine

Given the cosine double angle identity, use $\theta/2$ for θ and θ for $2 \cdot \theta$, then substitute the right angle identity for $\cos^2(\theta)$

$$\cos(\theta) = \cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right) = 1 - 2 \cdot \sin^2\left(\frac{\theta}{2}\right)$$

Subtract 1, divide by 2, and take the square root

Proof of Cosine

Given the cosine double angle identity, use $\theta/2$ for θ and θ for $2 \cdot \theta$, then substitute the right angle identity for $\sin^2(\theta)$

$$\cos(\theta) = \cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right) = 2 \cdot \cos^2\left(\frac{\theta}{2}\right) - 1$$

Add 1, divide by 2, and take the square root

Proof of Tangent

Divide the sine half angle identity by the cosine half angle identity, factor the root, then simplify

$$\left|\sin\left(\frac{\theta}{2}\right)\right| \Big/ \left|\cos\left(\frac{\theta}{2}\right)\right| = \sqrt{\frac{1 - \cos(\theta)}{2} \Big/ \frac{1 + \cos(\theta)}{2}} \rightarrow \left|\tan\left(\frac{\theta}{2}\right)\right| = \sqrt{\frac{1 - \cos(\theta)}{1 + \cos(\theta)}}$$

Proof of Cotangent

Divide the cosine half angle identity by the sine half angle identity, factor the root, then simplify

$$\left|\cos\left(\frac{\theta}{2}\right)\right| \Big/ \left|\sin\left(\frac{\theta}{2}\right)\right| = \sqrt{\frac{1 + \cos(\theta)}{2} \Big/ \frac{1 - \cos(\theta)}{2}} \rightarrow \left|\cot\left(\frac{\theta}{2}\right)\right| = \sqrt{\frac{1 + \cos(\theta)}{1 - \cos(\theta)}}$$

Square Identities

$$\sin^2(\theta) = \frac{1 - \cos(2 \cdot \theta)}{2} \qquad \cos^2(\theta) = \frac{1 + \cos(2 \cdot \theta)}{2} \qquad \tan^2(\theta) = \frac{1 - \cos(2 \cdot \theta)}{1 + \cos(2 \cdot \theta)} \qquad \cot^2(\theta) = \frac{1 + \cos(2 \cdot \theta)}{1 - \cos(2 \cdot \theta)}$$

Proofs

Given each of the half angle identities, use θ for $\theta/2$ and $2 \cdot \theta$ for θ, then square the equations

Polar ↔ Rectangular Conversion

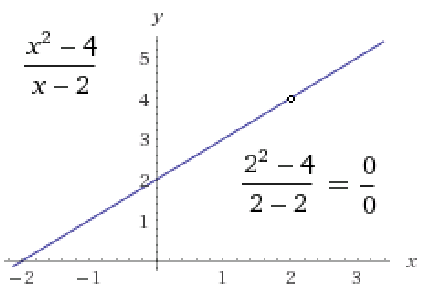
Given a Cartesian point (x, y) at polar coordinates with magnitude and angle (r, θ), the representations "transform" as such:

$$(r, \theta) = \left(\sqrt{x^2 + y^2}, \tan^{-1}\left(\frac{y}{x}\right)\right), x \neq 0 \qquad (x, y) = (r \cdot \sin(\theta), r \cdot \cos(\theta)) \in \mathbb{R}$$

Limits and Derivatives

Limits Definition and Properties

The output value a function approaches for a given input value. It is often used to determine undefined values in the function.



$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \frac{(x + 2)(x - 2)}{x - 2} = x + 2 = 4$$

Notation

- $\lim_{x \to a^+} f(x)$ represents the limit as the function approaches a point from the right.
- $\lim_{x \to a^-} f(x)$ represents the limit as the function approaches a point from the left.
- $\lim_{x \to a} f(x)$ represents the limit as the function approaches a point from both sides.

Continuity

A function $f(x)$ is continuous at point a if the limit exists and $f(a)$ is defined, in other words:

- The limit exists when $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) \equiv \lim_{x \to a} f(x)$
- There is no change in functionality, meaning there are no corners at $f(a)$
- $\lim_{x \to a} f(x) = f(a)$

Jump to Limits of Trig Functions
Jump to Differentiability
Jump to L'Hôpital's Rule

Limits Common Rules

Arithmetic Properties

$$\lim_{x \to a} c = c \qquad \lim_{x \to a}\big(f(x) \pm g(x)\big) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$$

$$\lim_{x \to a} c \cdot f(x) = c \cdot \lim_{x \to a} f(x) \qquad \lim_{x \to a} f(x) \cdot g(x)^{\pm 1} = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)^{\pm 1}$$

Jump to Derivative Sum and Difference Rules
Jump to Derivative Product Rule
Jump to Chain Rule
Jump to Derivative of an Inverse Function
Jump to L'Hôpital's Rule

Limits of Lines and Asymptotes

- Equation of a line: $\lim_{x \to a} f(x) = f(a) = m \cdot a + b$
- Horizontal asymptote: The line $x = c$ is defined as $\lim_{x \to c} f(x) = \pm\infty$
- Vertical asymptote: The line $y = c$ is defined as $\lim_{x \to \pm\infty} f(x) = c$

Squeeze Theorem

Given the transitive property and $f(x) \leq g(x) \leq h(x)$,

$$\text{If } \lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L, \text{then } \lim_{x \to a} g(x) = L$$

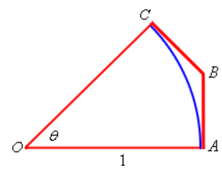
Limits of Trig Functions

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1 \qquad \lim_{x \to 0} \frac{\cos(x) - 1}{x} = 0$$

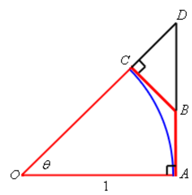
Jump to Derivatives of Trig Functions

Proof of Sine Limit

Using the incircle of a regular octagon with an apothem of magnitude 1 to define points in the sector for $0 \leq \theta \leq \pi/4$



Add to the top right corner a triangular region, forming an overall right triangle



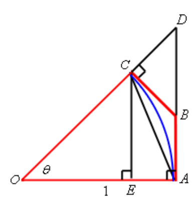
The following determination can be made inside of triangle *OAD*

$$\text{arc } AC < |AB| + |BC| < |AB| + |BD| = |AD| = |OA| \cdot \tan(\theta) = \tan(\theta) \ \rightarrow \ \text{arc } AC < \tan(\theta)$$

With the apothem being 1, the arc length *AC* being the angle θ. Substitute tangent for its cofunctions, then isolate $\cos(\theta)$.

$$\theta < \frac{\sin(\theta)}{\cos(\theta)} \ \rightarrow \ \cos(\theta) < \frac{\sin(\theta)}{\theta}$$

Connect point *C* perpendicularly to $|OA|$ at point E, and connect a line segment to $|AC|$



Note that $|CE| = |OC| \cdot \sin(\theta) = \sin(\theta)$ and $|CE| < |AC| < \text{arc } AC$. It follows that $\sin(\theta) < \theta$. Isolate 1.

$$\frac{\sin(\theta)}{\theta} < 1$$

Combine the two inequalities involving $\sin(\theta)/\theta$

$$\cos(\theta) < \frac{\sin(\theta)}{\theta} < 1$$

Given that $\lim_{\theta \to 0^+} \cos(\theta) = 1$ and $\lim_{\theta \to 0^+} 1 = 1$, use the squeeze theorem to determine the limit for $\sin(\theta)/\theta$

$$\lim_{\theta \to 0^+} \frac{\sin(\theta)}{\theta} = 1$$

Given that sine is an odd function, the limit from the other direction may be determined, therefore the limit exists

$$\frac{\sin(-\theta)}{-\theta} = \frac{-\sin(\theta)}{-\theta} = \frac{\sin(\theta)}{\theta} \therefore \lim_{\theta\to 0^-}\frac{\sin(\theta)}{\theta} = \lim_{\theta\to 0^+}\frac{\sin(\theta)}{\theta} = \lim_{\theta\to 0}\frac{\sin(\theta)}{\theta} = 1$$

Proof of Cosine Limit

Given the cosine limit, multiply by the following term over itself, simplify, and apply the right angle identity for sine

$$\lim_{\theta\to 0}\frac{\cos(\theta)-1}{\theta} = \lim_{\theta\to 0}\frac{\cos(\theta)-1}{\theta}\cdot\frac{\cos(\theta)+1}{\cos(\theta)+1} = \lim_{\theta\to 0}\frac{\cos^2(\theta)-1}{\theta\cdot(\cos(\theta)+1)} = \lim_{\theta\to 0}\frac{-\sin^2(\theta)}{\theta\cdot(\cos(\theta)+1)}$$

Isolate $\sin(\theta)/\theta$, then evaluate the limits

$$\lim_{\theta\to 0}\frac{-\sin(\theta)}{\cos(\theta)+1}\cdot\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta} = \frac{0}{2}\cdot 1 = 0$$

Limits Related to Zero and Infinity

$$\lim_{x\to 0^+}\frac{c}{x} = \infty, c \neq 0 \qquad \lim_{x\to 0^-}\frac{c}{x} = -\infty, c \neq 0 \qquad \lim_{x\to \pm\infty}\frac{c}{x} = 0, c \neq 0$$

Mathematical Constants

Proof of π

In regular polygons, as the number of sides $\to \infty$, the perimeter approaches a circumference, wherein PI can be calculated.

The Natural Number e

Continually compounded growth with 100% (1) return at a continuous rate yields a limit at e.

$$e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to 0}(1+x)^{1/x} \approx 2.7\ 1828\ 1828\ 459$$

Jump to Derivative of e Exponentiated
Jump to Analytic Trig Inverses

e Exponential Limit

Rearrangement of the second equality yields a commonly used limit

$$\lim_{x\to 0}\frac{e^x-1}{x} = 1$$

Natural Logarithm

The inverse function of e^x is $\log_e x$, represented as $\ln(x)$

Limits of Natural Exponents and Logarithms

$$\lim_{x\to\pm\infty} e^{\pm x} = \infty \qquad \lim_{x\to\pm\infty} e^{\mp x} = 0 \qquad \lim_{x\to\infty}\ln(x) = \infty \qquad \lim_{x\to 0^+}\ln(x) = -\infty$$

The Golden Ratio φ

Definition

Two quantities are in the golden ratio if their ratio is the same as the ratio of their sum to the larger of the two quantities

$$\frac{x+a}{x} = \frac{x}{a}$$

Proof of Ratios

Given the equality, substitute 1 for a, then multiply by x, and rearrange to appear as a quadratic equation

$$x+1 = x^2 \ \to\ x^2 - x - 1 = 0$$

Use the quadratic formula to find the values of x

$$x = \frac{-(-1)\pm\sqrt{(-1)^2-4\cdot 1\cdot(-1)}}{2\cdot 1} = \frac{1\pm\sqrt{5}}{2}$$

The solutions are typically represented as follows:

$$\varphi = \frac{1+\sqrt{5}}{2} \approx 1.61803 \qquad \phi = \frac{1-\sqrt{5}}{2} \approx -0.61803$$

Note that given the golden ratio equality, when expanding into a proper fraction and rearranging, it yields the value for ϕ

$$1 + \frac{1}{x} = x \ \rightarrow \ 1 - \varphi = -\frac{1}{\varphi} = \phi$$

The Fibonacci Series and φ Limit

The Fibonacci series is a recursive function whose numbers are the sum of its previous two numbers, starting with 1, 1

$$F_n = F_{n-1} + F_{n-2} = \{1,1,2,3,5,8,13,21,34,55,89,144 \dots\}$$

It can be rearranged as to yield an alternating list of values preceding the original function

$$F_{n-2} = F_n - F_{n-1} = \{\dots - 144, +89, -55, +34, -21, +13, -8, +5, -3, +2, -1, +1,0,1,1 \dots\}$$

As a number in the Fibonacci series becomes larger, the number divided by its previous number approaches the limits

$$\lim_{x \to \pm\infty} \frac{F(x)}{F(x-1)} = \pm\varphi^{\pm 1}$$

Golden Ratio Powers

Similar to the rearrangement of the quadratic of the equality, $x^2 = x + 1$, and given the numeric series, the following holds

$$\varphi^x = \varphi^{x-1} + \varphi^{x-2} = F(x) \cdot \varphi + F(x-1)$$

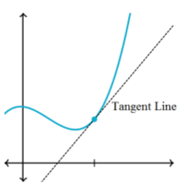
Algebraic Functions for the Fibonacci Series

$$F(x) = \frac{\varphi^x - (1-\varphi)^x}{\sqrt{5}}, x \in \mathbb{Z} \qquad F(x) = \frac{\varphi^x - \cos(\pi \cdot x) \cdot \varphi^{-x}}{\sqrt{5}}, x \in \mathbb{R}$$

Differentiation

Derivative Definition and Function

A function's instantaneous rate of change along any given point. The function is the slope formula with essentially one point.



$$\frac{d}{dx} f(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Jump to L'Hôpital's Rule

Notation

Derivative	Leibniz	Lagrange	Newton	Euler
First	$\frac{dy}{dx}$ or $\frac{d}{dx} f(x)$	$f'(x)$	$\dot{x}$	Df
Second	$\frac{d^2y}{dx^2}$ or $\frac{d^2}{dx^2} f(x)$	$f''(x)$	$\ddot{x}$	$D^2 f$

Tangent Line

The derivative form of the point-slope equation, which yields a line parallel to the function at one point it also touches

$$y - f(a) = f'(a)(x - a)$$

Differentiability

A function $f(x)$ is differentiable at a point a except if any of the following conditions is true

- $f(x)$ is not continuous at *a*
- $f(x)$ has a corner (change in function) at *a*
- $f(x)$ has a vertical tangent at *a*

First Derivative Use

- If $f'(a) = 0$, then $f(a)$ is a critical point (local maxima or minima)
- If $f'(a) > 0$, then $f(a)$ is increasing
- If $f'(a) < 0$, then $f(a)$ is decreasing
- If $f'(x)$ doesn't change signs, then $f(x)$ is monotonic (only increasing or decreasing)

Jump to Complex Number System

Second Derivative Use

- If $f''(a) = 0$, then $f(a)$ is an inflection point (change in concavity)
- If $f''(a) > 0$, then $f(a)$ is convex (concave upwards)
- If $f''(a) < 0$, then $f(a)$ is concave (downwards)

Derivatives Common Rules

Constant Rule

$$\frac{d}{dx}(C) = 0$$

A constant has a slope of zero at every point, therefore its derivative is zero at every point.

Constant Multiple Rule

$$\frac{d}{dx}(C \cdot f(x)) = C \cdot \frac{d}{dx} f(x)$$

Proof

$$\lim_{\Delta x \to 0} \frac{C \cdot f(x + \Delta x) - C \cdot f(x)}{\Delta x} = C \cdot \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Sum and Difference Rules

$$\frac{d}{dx}(f(x) \pm g(x)) = f'(x) \pm g'(x)$$

Proof

Insert the sum and difference expression into the derivative function

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) \pm g(x + \Delta x) - (f(x) \pm g(x))}{\Delta x}$$

Group the function in terms of f and g, then use the limit of sums property

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \pm \lim_{\Delta x \to 0} \frac{g(x + \Delta x) - g(x)}{\Delta x}$$

Rewrite the separate terms in derivative notation

Product Rule

$$\frac{d}{dx}(f(x) \cdot g(x)) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

Jump to Derivative of a Variable Raised to Itself
Jump to Complex Number System

Proof

Insert the product expression into the derivative function

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) \cdot g(x + \Delta x) - f(x) \cdot g(x)}{\Delta x}$$

Insert in the numerator a term equal to zero for the purpose of separating the fraction

$$\lim_{\Delta x\to 0}\frac{f(x+\Delta x)\cdot g(x+\Delta x)+f(x+\Delta x)\cdot g(x)-f(x+\Delta x)\cdot g(x)-f(x)\cdot g(x)}{\Delta x}$$

Separate the fraction, and apply the limit of sums property

$$\lim_{\Delta x\to 0}\left(f(x+\Delta x)\cdot\frac{g(x+\Delta x)-g(x)}{\Delta x}\right)+\lim_{\Delta x\to 0}\left(g(x)\cdot\frac{f(x+\Delta x)-f(x)}{\Delta x}\right)$$

Rewrite the terms in derivative notation and solve the remaining limit

Quotient Rule

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)=\frac{g(x)\cdot f'(x)-f(x)\cdot g'(x)}{g(x)^2}$$

Jump to Derivatives of Trig Functions

Proof

Insert the quotient expression into the derivative function, and factor

$$\lim_{\Delta x\to 0}\frac{1}{\Delta x}\cdot\left(\frac{f(x+\Delta x)}{g(x+\Delta x)}-\frac{f(x)}{g(x)}\right)=\lim_{\Delta x\to 0}\frac{1}{\Delta x}\cdot\frac{f(x+\Delta x)\cdot g(x)-f(x)\cdot g(x+\Delta x)}{g(x)\cdot g(x+\Delta x)}$$

Insert in the numerator a sum equal to zero for the purpose of separating the fraction

$$\lim_{\Delta x\to 0}\frac{f(x+\Delta x)\cdot g(x)-f(x)\cdot g(x)+f(x)\cdot g(x)-f(x)\cdot g(x+\Delta x)}{g(x)\cdot g(x+\Delta x)\cdot\Delta x}$$

Separate the fraction, distributing the negative for the second term

$$\lim_{\Delta x\to 0}\frac{f(x+\Delta x)\cdot g(x)-f(x)\cdot g(x)}{g(x)\cdot g(x+\Delta x)\cdot\Delta x}-\lim_{\Delta x\to 0}\frac{f(x)\cdot g(x+\Delta x)-f(x)\cdot g(x)}{g(x)\cdot g(x+\Delta x)\cdot\Delta x}$$

Reorganize to isolate the derivative functions

$$\lim_{\Delta x\to 0}\left(\frac{1}{g(x+\Delta x)}\cdot\frac{f(x+\Delta x)-f(x)}{\Delta x}\right)-\lim_{\Delta x\to 0}\left(\frac{f(x)}{g(x)\cdot g(x+\Delta x)}\cdot\frac{g(x+\Delta x)-g(x)}{\Delta x}\right)$$

Rewrite in terms of derivative notation, solve the remaining limits, and factor

$$\frac{f'(x)}{g(x)}-\frac{f(x)}{g(x)^2}\cdot g'(x)=\frac{f'(x)\cdot g(x)-f(x)\cdot g'(x)}{g(x)^2}$$

Power Rule

$$\frac{d}{dx}x^n=n\cdot x^{n-1},\forall n\in\mathbb{R}$$

Proof

Insert the power expression into the derivative function

$$\frac{d}{dx}x^n=\lim_{\Delta x\to 0}\frac{(x+\Delta x)^n-x^n}{\Delta x}$$

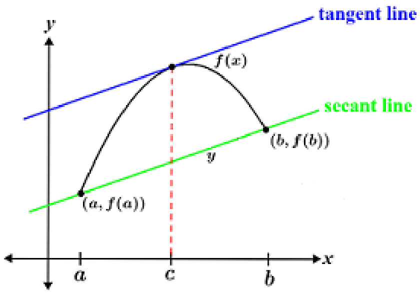
Apply binomial expansion, then combine like terms to cancel x^n

$$\frac{d}{dx}x^n=\lim_{\Delta x\to 0}\frac{\binom{n}{1}\cdot x^{n-1}\cdot\Delta x+\binom{n}{2}\cdot x^{n-2}\cdot\Delta x^2+\cdots+\Delta x^n}{\Delta x}$$

Separate the fraction to isolate the first term and cancel $\Delta x/\Delta x$ from it, solve the limit, then simplify the binomial coefficient

Mean Value Theorem for Derivatives

Within $[a,b]$, there exists a point c such that $f'(c)$ is equal to the function's average rate of change over (a,b)



$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Implicit Differentiation (Chain Rule)

The chain rule is the derivative for composite functions, meaning one function is within the other

Lagrange notation: $\frac{d}{dx} f\big(g(x)\big) = f'\big(g(x)\big) \cdot g'(x)$ Leibniz notation: $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

Jump to Derivative of Exponential Functions
Jump to Derivative of a Variable Raised to Itself
Jump to Derivatives of Trig Inverses

Proof

Recall that a derivative is a limit to an infinitesimal change, and that the limit of a product is the product of the limits

$$\frac{d}{dx} f\big(u(x)\big) = \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}$$

As $\Delta x \to 0$, $\Delta u \to 0$, therefore $\Delta x = \Delta u$. Solve the limits.

$$\frac{d}{dx} f\big(u(x)\big) = \frac{dy}{dx} = \lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = f'\big(g(x)\big) \cdot g'(x)$$

Derivative of an Inverse Function

Lagrange notation: $(f^{-1})'\big(f(x)\big) = \frac{1}{f'(x)}, f'(x) \neq 0$ Leibniz notation: $\frac{dg}{dx} = \left(\frac{df\big(g(x)\big)}{dg}\right)^{-1}$

Jump to Derivative of Logarithmic Functions

Proof

Insert the inverse expression into the derivative function, using $y = f(x)$, then factor $y - y$ into the denominator

$$(f^{-1})'(y) = \lim_{\Delta y \to 0} \frac{f^{-1}(y + \Delta y) - f^{-1}(y)}{y + \Delta y - y}$$

Given $y = f(x)$, and therefore $f^{-1}(y) = x$, rewrite the derivative function in terms of x

$$(f^{-1})'(y) = \lim_{\Delta y \to 0} \frac{x + \Delta x - x}{f(x + \Delta x) - f(x)}$$

Cancel like terms, rewrite the fraction, and simplify using the arithmetic properties of limits

$$(f^{-1})'(y) = 1 \Big/ \left(\lim_{\Delta y \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \right) = \frac{1}{f'(x)}$$

L'Hôpital's Rule

$$\text{If } \lim_{x \to a} f(x) = \lim_{x \to a} g(x) = (0 \text{ or } \pm\infty), \text{then } \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

The rule is specifically aimed at solving for indeterminate forms

The indeterminate form $0 \cdot \infty$

The rule holds when applying the reciprocal rule of division $(x \cdot y \to y/(1/x)$, also given that 0 becomes ∞ or ∞ becomes 0

The indeterminate form $\infty - \infty$

The rule holds when factoring terms into one fraction, given that infinity is the result of division by zero

Exponential Indeterminate Forms

In the case of $\lim_{x \to a} f(x)^{g(x)} = (1^\infty \text{ or } 0^0 \text{ or } \infty^0)$, take the natural logarithm and equate it to another limit L, then antilog it

$$\lim_{x \to a} g(x) \cdot \ln(f(x)) = \lim_{x \to a} \frac{\ln(f(x))}{1/g(x)} = \lim_{x \to a} \frac{g(x)}{1/\ln(f(x))} = L \qquad \lim_{x \to a} f(x)^{g(x)} = e^L$$

Proof

Let $f'(x)$ and $g'(x)$ be continuous at a and $f(a) = g(a) = 0$ such that

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)}$$

Apply the standard mean value theorem version of the derivative function, use the limits of quotients property, and simplify

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = \left(\lim_{x \to a} \frac{f(x) - f(a)}{x - a}\right) \Big/ \left(\lim_{x \to a} \frac{g(x) - g(a)}{x - a}\right) = \lim_{x \to a} \frac{f(x) - f(a)}{g(x) - g(a)}$$

Evaluate the limit. The same reasoning applies for ∞/∞ when applying the reciprocal rule of division.

$$\frac{\infty}{\infty} = \frac{f}{g} \to \frac{1/g}{1/f} = \frac{0}{0}$$

Derivative of *e* Exponentiated

$$\frac{d}{dx} e^x = e^x$$

Jump to Complex Number System

Proof

Insert the exponential expression into the derivative function, apply the exponents power rule, then factor e^x from the limit

$$\frac{d}{dx} e^x = \lim_{\Delta x \to 0} \frac{e^{x + \Delta x} - e^x}{\Delta x} = \lim_{\Delta x \to 0} \frac{e^x \cdot e^{\Delta x} - e^x}{\Delta x} = e^x \cdot \lim_{\Delta x \to 0} \frac{(e^{\Delta x} - 1)}{\Delta x}$$

The resulting limit is the *e* exponential limit equal to one

Derivative of Logarithmic Functions

$$\frac{d}{dx} = \log_b x = (x \cdot \ln(b))^{-1}, x > 0 \wedge b \neq 1 \qquad \frac{d}{dx} \ln(x) = x^{-1}, x > 0$$

Proof

Given that e^x and $\ln(x)$ are inverses of each other, insert the values into the inverse function derivative, and simplify

$$\frac{d}{dx} \ln(x) = \frac{1}{e^{\ln(x)}} = \frac{1}{x}$$

Differentiate the logarithms change of base formula (to e), extract the constant, and derive

$$\frac{d}{dx} \log_b x = \frac{d}{dx} \frac{\ln(x)}{\ln(b)} = \frac{1}{\ln(b)} \cdot \frac{d}{dx} \ln(x) = \frac{1}{x \cdot \ln(b)}$$

Derivative of Exponential Functions

$$\frac{d}{dx} b^x = b^x \cdot \ln(b)$$

Proof

Let $y = b^x$, so that $\ln(y) = \ln(b^x)$, use the logarithms power rule, then derive

$$\frac{d}{dx} \ln(y) = \frac{d}{dx} x \cdot \ln(b)$$

Apply the chain rule on the left, then evaluate the derivative on the right

$$y^{-1}\frac{dy}{dx} = \ln(b)$$

Multiply the equation by y, and substitute y for b^x

Derivative of a Variable Raised to Itself

$$\frac{d}{dx}x^x = x^x \cdot (1 + \ln(x))$$

Proof

Let $y = x^x$ so that $\ln(y) = \ln(x^x)$, use the logarithms power rule, then derive

$$\frac{d}{dx}\ln(y) = \frac{d}{dx}x \cdot \ln(x)$$

Define components of the product rule

$f(x) = x$ $\quad$ $f'(x) = 1$ $\quad$ $g(x) = \ln(x)$ $\quad$ $g'(x) = 1/x$

Derive, using the chain rule on the left and the product rule on the right, then simplify

$$\frac{1}{y}\cdot\frac{dy}{dx} = x \cdot \frac{1}{x} + \ln(x) \cdot 1 = 1 + \ln(x)$$

Multiply both sides by y and substitute y for its terms in x

Trigonometric Derivatives

Derivatives of Trig Functions

$\frac{d}{dx}\sin(x) = \cos(x)$ $\quad$ $\frac{d}{dx}\tan(x) = \sec^2(x)$ $\quad$ $\frac{d}{dx}\sec(x) = \sec(x) \cdot \tan(x), \cos(x) \neq 0$

$\frac{d}{dx}\cos(x) = -\sin(x)$ $\quad$ $\frac{d}{dx}\cot(x) = -\csc^2(x)$ $\quad$ $\frac{d}{dx}\csc(x) = -\csc(x) \cdot \cot(x), \sin(x) \neq 0$

Jump to Complex Number System

Higher Order Sine and Cosine Derivatives

The sine and cosine derivatives alternate and change signs as they are continued, therefore the following holds:

$\frac{d^{2\cdot n}y}{dx^{2\cdot n}}\sin(x) = (-1)^n \cdot \sin(x)\ \forall n \in \mathbb{N}$ $\quad$ $\frac{d^{2\cdot n}y}{dx^{2\cdot n}}\cos(x) = (-1)^n \cdot \cos(x)\ \forall n \in \mathbb{N}$

Proof of Sine

Insert $\sin(x)$ into the derivative function, and expand the sine addition term

$$\lim_{\Delta x \to 0}\frac{\sin(x + \Delta x) - \sin(x)}{\Delta x} = \lim_{\Delta x \to 0}\frac{\sin(\Delta x) \cdot \cos(x) + \sin(x) \cdot \cos(\Delta x) - \sin(x)}{\Delta x}$$

Factor into terms of limits of trig functions, solve the limits, and simplify

$$\lim_{\Delta x \to 0}\left(\frac{\sin(\Delta x)}{\Delta x} \cdot \cos(x) + \frac{\cos(\Delta x) - 1}{\Delta x} \cdot \sin(x)\right) = 1 \cdot \cos(x) + 0 \cdot \sin(x) = \cos(x)$$

Proof of Cosine

Insert $cos(x)$ into the derivative function, and expand the cosine addition term

$$\lim_{\Delta x \to 0}\frac{\cos(x + \Delta x) - \cos(x)}{\Delta x} = \lim_{\Delta x \to 0}\frac{\cos(x) \cdot \cos(\Delta x) - \sin(x) \cdot \sin(\Delta x) - \cos(x)}{\Delta x}$$

Factor into terms of limits of trig functions, solve the limits, and simplify

$$\lim_{\Delta x \to 0}\left(\frac{\cos(\Delta x)-1}{\Delta x}\cdot\cos(x)-\frac{\sin(\Delta x)}{\Delta x}\cdot\sin(x)\right)=0\cdot\cos(x)-1\cdot\sin(x)=-\sin(x)$$

Proof of Tangent

Write in terms of sine and cosine, then use the quotient rule, and simplify

$$\frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right)=\frac{\cos(x)\cdot\frac{d}{dx}\sin(x)-\sin(x)\cdot\frac{d}{dx}\cos(x)}{\cos^2(x)}=\frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)}$$

Substitute the right angle identity and rewrite cosine in terms of secant

Proof of Cotangent

Write in terms of sine and cosine, then use the quotient rule, and simplify

$$\frac{d}{dx}\left(\frac{\cos(x)}{\sin(x)}\right)=\frac{\sin(x)\cdot\frac{d}{dx}\cos(x)-\cos(x)\cdot\frac{d}{dx}\sin(x)}{\sin^2(x)}=-\frac{\sin^2(x)+\cos^2(x)}{\sin^2(x)}$$

Substitute the right angle identity and rewrite sine in terms of cosecant

Proof of Secant

Rewrite in cosine form, use the quotient rule, simplify, and substitute the cofunctions

$$\frac{d}{dx}\left(\frac{1}{\cos(x)}\right)=\frac{\cos(x)\cdot 0-1\cdot(-\sin(x))}{\cos^2(x)}=\frac{\sin(x)}{\cos^2(x)}=\frac{1}{\cos(x)}\cdot\frac{\sin(x)}{\cos(x)}$$

Proof of Cosecant

Rewrite in sine form, use the quotient rule, simplify, and substitute the cofunctions

$$\frac{d}{dx}\left(\frac{1}{\sin(x)}\right)=\frac{\sin(x)\cdot 0-1\cdot\cos(x)}{\sin^2(x)}=-\frac{\cos(x)}{\sin^2(x)}=-\frac{1}{\sin(x)}\cdot\frac{\cos(x)}{\sin(x)}$$

Derivatives of Trig Inverses

$$\frac{d}{dx}\sin^{-1}\left(\pm\frac{x}{a}\right)=\pm\frac{1}{\sqrt{a^2-x^2}},|x|<1 \qquad \frac{d}{dx}\tan^{-1}\left(\pm\frac{x}{a}\right)=\pm\frac{a}{x^2+a^2} \qquad \frac{d}{dx}\sec^{-1}\left(\pm\frac{x}{a}\right)=\pm\frac{a}{\sqrt{x^2\cdot(x^2-a^2)}},|x|>1$$

$$\frac{d}{dx}\cos^{-1}\left(\pm\frac{x}{a}\right)=\mp\frac{1}{\sqrt{a^2-x^2}},|x|<1 \qquad \frac{d}{dx}\cot^{-1}\left(\pm\frac{x}{a}\right)=\mp\frac{a}{x^2+a^2} \qquad \frac{d}{dx}\csc^{-1}\left(\pm\frac{x}{a}\right)=\mp\frac{a}{\sqrt{x^2\cdot(x^2-a^2)}},|x|>1$$

Proof of Inverse Sine

Let $y=\sin^{-1}(\pm x/a)$ so that $\sin(y)=\pm x/a$, then differentiate the second equality

$$\frac{d}{dx}\sin(y)=\pm\frac{1}{a}\cdot\frac{d}{dx}x$$

Use the chain rule on the left, then simplify on the right, and divide by $\cos(y)$

$$\cos(y)\cdot\frac{dy}{dx}=\pm\frac{1}{a}\rightarrow\frac{dy}{dx}=\pm\frac{1}{a\cdot\cos(y)}$$

Substitute the right angle identity for the cosine term. Because of the range of y, $\cos(y)\geq 0$ ∴ the square root is positive.

$$\frac{dy}{dx}=\pm\frac{1}{a\cdot\sqrt{1-\sin^2(y)}}$$

Substitute y and $\sin(y)$ for their original values, and simplify

$$\frac{d}{dx}\sin^{-1}\left(\pm\frac{x}{a}\right)=\pm\frac{1}{a\cdot\sqrt{1-x^2/a^2}}=\pm\frac{1}{\sqrt{a^2-x^2}}$$

Proof of Inverse Cosine

Let $y = \cos^{-1}(\pm x/a)$ so that $\cos(y) = \pm x/a$, then differentiate the second equality

$$\frac{d}{dx}\cos(y) = \pm\frac{1}{a}\cdot\frac{d}{dx}x$$

Use the chain rule on the left, then simplify on the right, and divide by $-\sin(y)$

$$-\sin(y)\cdot\frac{dy}{dx} = \pm\frac{1}{a} \rightarrow \frac{dy}{dx} = \mp\frac{1}{a\cdot\sin(y)}$$

Substitute the right angle identity for the sine term. Because of the range of y, $\sin(y) \geq 0 \therefore$ the square root is positive.

$$\frac{dy}{dx} = \mp\frac{1}{a\cdot\sqrt{1-\cos^2(x)}}$$

Substitute y and $\cos(y)$ for their original values, and simplify

$$\frac{d}{dx}\cos^{-1}\left(\pm\frac{x}{a}\right) = \mp\frac{1}{a\cdot\sqrt{1-x^2/a^2}} = \mp\frac{1}{\sqrt{a^2-x^2}}$$

Proof of Inverse Tangent

Let $y = \tan^{-1}(\pm x/a)$ so that $\tan(y) = \pm x/a$, then differentiate the second equality

$$\frac{d}{dx}\tan(y) = \pm\frac{1}{a}\cdot\frac{d}{dx}x$$

Use the chain rule on the left, then simplify on the right, and divide by $\sec^2(y)$

$$\sec^2(y)\cdot\frac{dy}{dx} = \pm\frac{1}{a} \rightarrow \frac{dy}{dx} = \pm\frac{1}{a\cdot\sec^2(y)}$$

Substitute the right angle identity for the secant term

$$\frac{dy}{dx} = \pm\frac{1}{a\cdot(\tan^2(y)+1)}$$

Substitute y and $\tan(y)$ for their original values, and simplify

$$\frac{d}{dx}\tan^{-1}\left(\pm\frac{x}{a}\right) = \pm\frac{1}{a\cdot(x^2/a^2+1)} = \pm\frac{1}{(x^2/a+a)} = \pm\frac{a}{x^2+a^2}$$

Proof of Inverse Cotangent

Let $y = \cot^{-1}(\pm x/a)$ so that $\cot(y) = \pm x/a$, then differentiate the second equality

$$\frac{d}{dx}\cot(y) = \pm\frac{1}{a}\cdot\frac{d}{dx}x$$

Use the chain rule on the left, then simplify on the right, and divide by $-\csc^2(y)$

$$-\csc^2(y)\cdot\frac{dy}{dx} = \pm\frac{1}{a} \rightarrow \frac{dy}{dx} = \mp\frac{1}{a\cdot\csc^2(y)}$$

Substitute the right angle identity for the cosecant term

$$\frac{dy}{dx} = \mp\frac{1}{a\cdot(\cot^2(y)+1)}$$

Substitute y and $\cot(y)$ for their original values, and simplify

$$\frac{d}{dx}\cot^{-1}\left(\pm\frac{x}{a}\right) = \mp\frac{1}{a\cdot(x^2/a^2+1)} = \mp\frac{1}{(x^2/a+a)} = \mp\frac{a}{x^2+a^2}$$

Proof of Inverse Secant

Let $y = \sec^{-1}(\pm x/a)$ so that $\sec(y) = \pm x/a$, then differentiate the second equality

$$\frac{d}{dx}\sec(y) = \pm\frac{1}{a}\cdot\frac{d}{dx}x$$

Use the chain rule on the left, then simplify on the right, and divide by $\sec(y)\cdot\tan(y)$

$$\sec(y)\cdot\tan(y)\cdot\frac{dy}{dx} = \pm\frac{1}{a} \rightarrow \frac{dy}{dx} = \pm\frac{1}{a\cdot\sec(y)\cdot\tan(y)}$$

Substitute the right angle identity for the tangent term, then factor terms into the radicand

$$\frac{dy}{dx} = \pm\frac{1}{a\cdot\sec(y)\cdot\sqrt{\sec^2(y)-1}} = \pm\frac{1}{\sqrt{a^2\cdot\sec^2(y)\cdot(\sec^2(y)-1)}}$$

Substitute y and $\sec(y)$ for their original values, and simplify

$$\frac{d}{dx}\sec^{-1}\left(\pm\frac{x}{a}\right) = \pm\frac{1}{\sqrt{a^2\cdot x^2/a^2\cdot(x^2/a^2-1)}} = \pm\frac{1}{\sqrt{x^2\cdot((x^2-a^2)/a^2)}} = \pm\frac{a}{\sqrt{x^2\cdot(x^2-a^2)}}$$

Proof of Inverse Cosecant

Let $y = \csc^{-1}(\pm x/a)$ so that $\csc(y) = \pm x/a$, then differentiate the second equality

$$\frac{d}{dx}\csc(y) = \pm\frac{1}{a}\cdot\frac{d}{dx}x$$

Use the chain rule on the left, then simplify on the right, and divide by $-\csc(y)\cdot\cot(y)$

$$-\csc(y)\cdot\cot(y)\cdot\frac{dy}{dx} = \pm\frac{1}{a} \rightarrow \frac{dy}{dx} = \mp\frac{1}{a\cdot\csc(y)\cdot\cot(y)}$$

Substitute the right angle identity for the cotangent term, then factor terms into the radicand

$$\frac{dy}{dx} = \mp\frac{1}{a\cdot\csc(y)\cdot\sqrt{\csc^2(y)-1}} = \mp\frac{1}{\sqrt{a^2\cdot\csc^2(y)\cdot(\csc^2(y)-1)}}$$

Substitute y and $\csc(y)$ for their original values, and simplify

$$\frac{d}{dx}\csc^{-1}\left(\pm\frac{x}{a}\right) = \mp\frac{1}{\sqrt{a^2\cdot x^2/a^2\cdot(x^2/a^2-1)}} = \mp\frac{1}{\sqrt{x^2\cdot((x^2-a^2)/a^2)}} = \mp\frac{a}{\sqrt{x^2\cdot(x^2-a^2)}}$$

Complex Analysis and Trigonometry

Complex Number System

Recall the sections on complex numbers and the complex unit circle

Euler's Formula	$e^{\pm i\cdot\theta} = \cos(\theta) \pm i\cdot\sin(\theta), \forall\theta \in \mathbb{R}$
Form Equivalences	$r\cdot i^{2\cdot x/\pi} = r\cdot e^{i\cdot x} = r\cdot(\cos(x) \pm i\cdot\sin(x)) = r\left(\frac{a}{r}+\frac{b}{r}\cdot i\right)$
Natural Logarithms of Negatives	$\ln(-x) = i\cdot\pi + \ln(x), \forall x > 0 \in \mathbb{R}$
Natural Logarithms of Imaginaries	$\ln(i) = \frac{\pi}{2}\cdot i$

Proof of Euler's Formula

Let $f(\theta) = e^{-i\cdot\theta}\cdot(\cos(\theta) + i\cdot\sin(\theta))$, then derive using the product rule, implicit exponential rule, and trig derivatives

$$\frac{d}{dx}f(\theta) = -i\cdot e^{-i\cdot\theta}\cdot(\cos(\theta) + i\cdot\sin(\theta)) + e^{-i\cdot\theta}\cdot(-\sin(\theta) + i\cdot\cos(\theta))$$

Distribute $-i$, factor out $e^{-i\cdot\theta}$, then cancel like terms

$$\frac{d}{dx}f(\theta) = e^{-i\cdot\theta}\cdot(-i\cdot\cos(\theta) + \sin(\theta) - \sin(\theta) + i\cdot\cos(\theta)) = e^{-i\cdot\theta}\cdot 0 = 0$$

Since $f'(\theta) = 0$, $f(\theta)$ is a constant. Set $\theta = 0$ to solve for c.

$$c = e^{-i\cdot\theta} \cdot (\cos(\theta) + i \cdot \sin(\theta)) = e^{-i\cdot 0} \cdot (\cos(0) + i \cdot \sin(0)) = e^{0} \cdot (1 + i \cdot 0) = 1$$

Divide by $e^{-i\cdot\theta}$

$$e^{-i\cdot\theta} \cdot (\cos(\theta) + i \cdot \sin(\theta)) = 1 \;\rightarrow\; \cos(\theta) + i \cdot \sin(\theta) = e^{i\cdot\theta}$$

Proof of Negative Natural Logarithms

Given $\ln(-x)$, factor negative one from the inside term, then apply the logarithms product rule

$$\ln(-1) + \ln(x)$$

Substitute the -1 using Euler's formula for $x = \pi$ (Euler's identity), and apply the logarithms base inverse property

$$\ln(e^{i\pi}) + \ln(x) = \; i \cdot \pi + \ln(x)$$

Proof of Imaginary Natural Logarithms

Given the real-polar relation, take the natural logarithm and cancel the variable

$$\ln(i^{x}) = \ln\left(e^{i\cdot\frac{\pi}{2}\cdot x}\right) \;\rightarrow\; x \cdot \ln(i) = i \cdot \frac{\pi}{2} \cdot x$$

Analytic Trig Functions

$$\sin(\theta) = \frac{e^{i\cdot\theta} - e^{-i\cdot\theta}}{2 \cdot i} \qquad \cos(\theta) = \frac{e^{i\cdot\theta} + e^{-i\cdot\theta}}{2} \qquad \tan(\theta) = \frac{e^{i\cdot\theta} - e^{-i\cdot\theta}}{i \cdot (e^{i\cdot\theta} + e^{-i\cdot\theta})}$$

$$\csc(\theta) = \frac{2 \cdot i}{e^{i\cdot\theta} - e^{-i\cdot\theta}} \qquad \sec(\theta) = \frac{2}{e^{i\cdot\theta} + e^{-i\cdot\theta}} \qquad \cot(\theta) = \frac{i \cdot \left(e^{i\cdot\theta} + e^{-i\cdot\theta}\right)}{e^{i\cdot\theta} - e^{-i\cdot\theta}}$$

Proof of Sine and Cosine

Given Euler's formula, add to it the negative version of itself, cancel like terms, then divide by 2 for the cosine function

$$e^{i\cdot\theta} + e^{-i\cdot\theta} = \cos(\theta) + i \cdot \sin(\theta) + \cos(\theta) - i \cdot \sin(\theta) = 2 \cdot \cos(\theta) \;\rightarrow\; \cos(\theta) = \frac{e^{i\cdot\theta} + e^{-i\cdot\theta}}{2}$$

Subtract from Euler's formula the negative version of itself, cancel like terms, then divide by $2 \cdot i$ for the sine function

$$e^{i\cdot\theta} - e^{-i\cdot\theta} = \cos(\theta) + i \cdot \sin(\theta) - \cos(\theta) + i \cdot \sin(\theta) = 2 \cdot i \cdot \sin(\theta) \;\rightarrow\; \sin(\theta) = \frac{e^{i\cdot\theta} - e^{-i\cdot\theta}}{2 \cdot i}$$

Proof of Tangent, Cotangent, Secant, and Cosecant

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{e^{i\cdot\theta} - e^{-i\cdot\theta}}{i \cdot (e^{i\cdot\theta} + e^{-i\cdot\theta})} \qquad \sec(\theta) = \frac{1}{\cos(\theta)} = \frac{2}{e^{i\cdot\theta} + e^{-i\cdot\theta}}$$

$$\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)} = \frac{i \cdot \left(e^{i\cdot\theta} + e^{-i\cdot\theta}\right)}{e^{i\cdot\theta} - e^{-i\cdot\theta}} \qquad \csc(\theta) = \frac{1}{\sin(\theta)} = \frac{2 \cdot i}{e^{i\cdot\theta} - e^{-i\cdot\theta}}$$

Analytic Trig Inverses

$$\sin^{-1}(\theta) = -i \cdot \ln\left(i \cdot \theta + \sqrt{1 - \theta^2}\right) \qquad \cos^{-1}(\theta) = -i \cdot \ln\left(\theta + i \cdot \sqrt{1 - \theta^2}\right) \qquad \tan^{-1}(\theta) = \frac{i}{2} \cdot \ln\left(\frac{1 - i \cdot \theta}{1 + i \cdot \theta}\right)$$

$$\csc^{-1}(\theta) = -i \cdot \ln\left(i \cdot \theta^{-1} + \sqrt{1 - \theta^{-2}}\right) \qquad \sec^{-1}(\theta) = -i \cdot \ln\left(\theta^{-1} + i \cdot \sqrt{1 - \theta^{-2}}\right) \qquad \cot^{-1}(\theta) = \frac{i}{2} \cdot \ln\left(\frac{i \cdot \theta + 1}{i \cdot \theta - 1}\right)$$

Proof of Inverse Sine

Finding the inverse for $\sin(\theta)$ indicates that $\theta \rightarrow \sin^{-1}(\theta)$ within the function and $\sin(\theta) \rightarrow \theta$

$$\theta = \frac{e^{i\cdot\sin^{-1}(\theta)} - e^{-i\cdot\sin^{-1}(\theta)}}{2 \cdot i}$$

Let $z = e^{i\cdot\sin^{-1}(\theta)}$, multiply by z, then rearrange into quadratic form

$$0 = z^2 - 2 \cdot i \cdot \theta \cdot z - 1$$

Use the quadratic formula to determine the values of z, then simplify

$$z = \frac{2 \cdot i \cdot \theta \pm \sqrt{(2 \cdot i \cdot \theta)^2 + 4}}{2} = i \cdot \theta \pm \sqrt{1 - \theta^2}$$

Substitute z for its original value, then take the natural logarithm, and divide by i

$$e^{i \cdot \sin^{-1}(\theta)} = i \cdot \theta \pm \sqrt{1 - \theta^2} \rightarrow \sin^{-1}(\theta) = -i \cdot \ln\left(i \cdot \theta \pm \sqrt{1 - \theta^2}\right)$$

The positive square root within the function yields the inverse sine function while the negative square root does not

Proof of Inverse Cosine

Finding the inverse for $\cos(\theta)$ indicates that $\theta \rightarrow \cos^{-1}(\theta)$ within the function and $\cos(\theta) \rightarrow \theta$

$$\theta = \frac{e^{i \cdot \cos^{-1}(\theta)} + e^{-i \cdot \cos^{-1}(\theta)}}{2}$$

Let $z = e^{i \cdot \cos^{-1}(\theta)}$, multiply by z, then rearrange into quadratic form

$$0 = z^2 - 2 \cdot \theta \cdot z + 1$$

Use the quadratic formula to determine the values of z, then simplify

$$z = \frac{2 \cdot \theta \pm \sqrt{(-2 \cdot \theta)^2 - 4}}{2} = \theta \pm \sqrt{-1 \cdot (1 - \theta^2)} = \theta \pm i \cdot \sqrt{(1 - \theta^2)}$$

Substitute z for its original value, then take the natural logarithm, and divide by i

$$e^{i \cdot \cos^{-1}(\theta)} = \theta \pm i \cdot \sqrt{(1 - \theta^2)} \rightarrow \cos^{-1}(\theta) = -i \cdot \ln\left(\theta \pm i \cdot \sqrt{(1 - \theta^2)}\right)$$

The positive square root within the function yields the inverse cosine function while the negative square root does not

Proof of Inverse Tangent

Finding the inverse for $\tan(\theta)$ indicates that $\theta \rightarrow \tan^{-1}(\theta)$ within the function and $\tan(\theta) \rightarrow \theta$

$$\theta = \frac{e^{i \cdot \tan^{-1}(\theta)} - e^{-i \cdot \tan^{-1}(\theta)}}{i\left(e^{i \cdot \tan^{-1}(\theta)} + e^{-i \cdot \tan^{-1}(\theta)}\right)}$$

Let $z = e^{i \cdot \tan^{-1}(\theta)}$, multiply by the denominator, then expand

$$\theta = \frac{z - z^{-1}}{i(z + z^{-1})} \rightarrow i \cdot \theta \cdot z + i \cdot \theta \cdot z^{-1} = z - z^{-1}$$

Combine like terms, multiply by $-z$, then factor z

$$z^2 \cdot (1 - i \cdot \theta) = 1 + i \cdot \theta$$

Isolate $1/z^2$, take the natural logarithm, substitute z for its original value, then distribute the exponent

$$\ln\left(\frac{1 - i \cdot \theta}{1 + i \cdot \theta}\right) = \ln\left(e^{-2 \cdot i \cdot \tan^{-1}(\theta)}\right)$$

Apply the logarithms base inverse property, and divide by $-2 \cdot i$

$$\ln\left(\frac{1 - i \cdot \theta}{1 + i \cdot \theta}\right) = -2 \cdot i \cdot \tan^{-1}(\theta) \rightarrow \tan^{-1}(\theta) = \frac{i}{2} \cdot \ln\left(\frac{1 - i \cdot \theta}{1 + i \cdot \theta}\right)$$

Proof of Inverse Cotangent

Finding the inverse for $\cot(\theta)$ indicates that $\theta \rightarrow \cot^{-1}(\theta)$ within the function and $\cot(\theta) \rightarrow \theta$

$$\theta = \frac{i \cdot \left(e^{i \cdot \cot^{-1}(\theta)} + e^{-i \cdot \cot^{-1}(\theta)}\right)}{e^{i \cdot \cot^{-1}(\theta)} - e^{-i \cdot \cot^{-1}(\theta)}}$$

Let $z = e^{i \cdot \cot^{-1}(\theta)}$, multiply by the denominator, then expand

$$\theta = \frac{i \cdot (z + z^{-1})}{z - z^{-1}} \rightarrow \theta \cdot z - \theta \cdot z^{-1} = i \cdot z + i \cdot z^{-1}$$

Combine like terms, multiply by z, then factor z

$$z^2 \cdot (\theta - i) = i + \theta$$

Isolate $1/z^2$, multiply by i/i, take the natural logarithm, substitute z for its original value, then distribute the exponent

$$\ln\left(\frac{i \cdot \theta + 1}{i \cdot \theta - 1}\right) = \ln\left(e^{-2 \cdot i \cdot \cot^{-1}(\theta)}\right)$$

Apply the logarithms base inverse property, and divide by $-2 \cdot i$

$$\ln\left(\frac{i \cdot \theta + 1}{i \cdot \theta - 1}\right) = -2 \cdot i \cdot \cot^{-1}(\theta) \rightarrow \cot^{-1}(\theta) = \frac{i}{2} \cdot \ln\left(\frac{i \cdot \theta + 1}{i \cdot \theta - 1}\right)$$

Proof of Inverse Secant

Finding the inverse for $\sec(\theta)$ indicates that $\theta \rightarrow \sec^{-1}(\theta)$ within the function and $\sec(\theta) \rightarrow \theta$

$$\theta = \frac{2}{e^{i \cdot \sec^{-1}(\theta)} + e^{-i \cdot \sec^{-1}(\theta)}}$$

Let $z = e^{i \cdot \sec^{-1}(\theta)}$, multiply by z, then rearrange into quadratic form

$$\theta \cdot z^2 - 2 \cdot z + \theta = 0$$

Use the quadratic formula to determine the values of z, then simplify

$$z = \frac{2 \pm \sqrt{(-2)^2 - 4 \cdot \theta^2}}{2 \cdot \theta} = \frac{1 \pm \sqrt{-1 \cdot (\theta^2 - 1)}}{\theta} = \theta^{-1} \pm i \cdot \sqrt{1 - \theta^{-2}}$$

Substitute z for its original value, then take the natural logarithm, and divide by i

$$e^{i \cdot \sec^{-1}(\theta)} = \theta^{-1} \pm i \cdot \sqrt{1 - \theta^{-2}} \rightarrow \sec^{-1}(\theta) = -i \cdot \ln\left(\theta^{-1} \pm i \cdot \sqrt{1 - \theta^{-2}}\right)$$

The positive square root within the function yields the inverse sine function while the negative square root does not

Proof of Inverse Cosecant

Finding the inverse for $\csc(\theta)$ indicates that $\theta \rightarrow \csc^{-1}(\theta)$ within the function and $\csc(\theta) \rightarrow \theta$

$$\theta = \frac{2 \cdot i}{e^{i \cdot \csc^{-1}(\theta)} - e^{-i \cdot \csc^{-1}(\theta)}}$$

Let $z = e^{i \cdot \csc^{-1}(\theta)}$, multiply by z, then rearrange into quadratic form

$$\theta \cdot z^2 - 2 \cdot i \cdot z - \theta = 0$$

Use the quadratic formula to determine the values of z, then simplify

$$z = \frac{2 \cdot i \pm \sqrt{(-2 \cdot i)^2 + 4 \cdot \theta^2}}{2 \cdot \theta} = \frac{i \pm \sqrt{\theta^2 - 1}}{\theta} = i \cdot \theta^{-1} \pm \sqrt{1 - \theta^{-2}}$$

Substitute z for its original value, then take the natural logarithm, and divide by i

$$e^{i \cdot \csc^{-1}(\theta)} = i \cdot \theta^{-1} \pm \sqrt{1 - \theta^{-2}} \rightarrow \csc^{-1}(\theta) = -i \cdot \ln\left(i \cdot \theta^{-1} \pm \sqrt{1 - \theta^{-2}}\right)$$

The positive square root within the function yields the inverse sine function while the negative square root does not